Contagion in Stable Networks

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Abstract

We study the formation of networks in environments where agents derive benefits from other agents directly linked to them but suffer losses through contagion when any agent on a path connected to them is hit by a shock. We first consider networks with undirected links (e.g. epidemics, underground resistance organizations, trade networks) where we find that stable networks are comprised of completely connected disjoint subnetworks. Then, we consider networks with directed links and we find that the completely connected network is stable, although, its exact structure, and thus contagion implications, is sensitive to parameter values for costs and benefits. Lastly, we introduce aggregate externalities (e.g. fire sales for the case of financial networks) and we find that stable networks can be asymmetric, connected but not completely connected, thus capturing the main features of inter-industry and financial networks.

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1. Introduction

A significant part of the literature on strategic network formation has focused on variants of the 'connections' model studied by Jackson and Wolinsky (1996). The common idea of this literature is that being part of a network allows agents to benefit not only from their direct links but also from indirect connections to other agents in the network. In contrast, the costs of network participation in these models are only associated with the creation of direct links. However, as Blume *et al.* (2011) observe in many networks studied in economics and other disciplines the structure of costs and benefits is inverted. As an example from economics they refer to the extensive body of work on issues related to systemic risk in financial networks. In those networks two institutions form a link by signing a loan agreement from which each party derives a benefit. A failure by one institution to meet its obligations inflicts costs not only to the two parties that have signed

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the agreement but also to other institutions connected to them, directly or indirectly, by other financial agreements.

The above observation motivated Blume *et al.* (2011) to study the formation of networks with this alternative general payoff structure. They have restricted their analysis to undirected graphs where information can flow in either direction along a link. Using stability as a solution concept that allows them to make predictions about which network structures are more likely to form they find stable networks that consist of fully connected disjoint subgraphs (cliques). Such arrangements might be offering a good description of some examples of social networks they mentioned in their paper (e.g. formation of groups that minimize the risk of disease epidemics and the organization of cladenstine operations) but definitely less so for financial systems that have network structures which are connected but incomplete.[1] In such networks there is always a path connecting any of the nodes (financial institutions) with every other node (ignoring for the moment that links can also be directed), however, not all nodes are directly linked with every other node.

In this paper, we argue that by distinguishing between directed and undirected graphs we can explain such variations in network structures. We begin by analyzing the formation of undirected networks in a variant of the Blume *et al.* (2011) model. As in their model (a) agents derive a benefit by forming a link, and (b) each agent fails independently with some fixed probability. The difference in the two models is related to the way the costs associated with such failures spread through the network. In their model after a failure each node with some probability becomes live and shocks can only be transmitted through live nodes. In our model all nodes are live and thus affected by the shock, however, the magnitude of the losses for each node depends on its distance from the one that initially failed. We confirm that when links are undirected there exist stable networks that consist of fully connected disjoint subgraphs but we also show that these are the only structures that can be stable.

Then, we turn our attention to directed networks where shocks can be only transmitted along directed paths. In financial networks an outgoing (ingoing) link means that the institution represented by the node is a borrower (lender). Thus, the link captures the flow of financial liabilities. By having only a subset of nodes being live the Blume *et al.* (2011) model mimics the transmission of shocks in directed networks. However, there is an important difference. In their model the paths are exogenously determined but in our model are equilibrium outcomes. The direction of the links are determined by the strategic decisions of agents. We find that the most likely stable structures are complete directed networks (tournaments). More interestingly, we find that small variations in the parameters of the model can significantly affect the vulnerability of the network to a failure of any randomly chosen node. In particular, we find that by either slightly decreasing the benefit derived from forming a link or slightly decreasing the cost associated with failure we can move from a stable network structure where there is only one agent whose failure would lead to the collapse of the whole network to a structure where all agents become critical.

The prediction that stable networks with directed links are connected is in agreement with our knowledge on the structure of financial networks our prediction that such networks

[1]For a variety of examples see the review article by Bougheas and Kirman (2015).

are complete is not. With that in mind we next consider a modified version of our model that allows for aggregate externalities. More specifically, we consider the case where the costs suffered by each agent following a failure is increasing in the number of agents being affected by the failure. Now we find that there are parameter values such that any stable network is connected but not complete.

In the following section, we describe a few examples of networks from economics and other disciplines, both undirected and directed, whose general structure is captured by our model.[2] We also offer examples of aggregate externalities that might generate cost structures similar to the one we introduced in the last version of our model.

1.1. Contagion in Social and Economic Networks

The first three are examples of undirected networks and the following two are examples of directed networks where, as many researchers have suggested in the past, externalities might be playing an important role.

Contagious diseases and group size As Blume *et al.* (2011) observe there is a trade-off related to the formation of social group. Larger clusters increase the benefits of participants, however, they also increase the risk of contagion. This trade-off is clearly illustrated in the study by Hamilton *et al.* (2007) of hunter-gatherer societies where they make the distinction between cohesive and disruptive forces in the process of group formation and among the latter they identify the spread of diseases.

Underground resistance networks This is another of the examples offered by Blume *et al.* (2011). Participants in such organizations benefit by working in groups but also there is a risk that the group might be infiltrated. Chai (1993) explores this trade-off in the context of groups that resist governments while Morselli *et al.* (2007) do the same for criminal networks.

Globalization and the international transmission of shocks The free movement of goods and services, intermediate inputs in production and financial capital can be welfare enhancing during times of prosperity but they also facilitate the transmission of regional shocks around the globe. This trade-off has been studied by Imbs (2004), Kose *et al.* (2003) and since then has been an active topic of research.

What all three examples mentioned above have in common is that links are symmetric and thus undirected.[3] In all three examples he size of the group is a source of tension between opposing forces. Larger groups confer benefits to participants as there are more opportunities for collaboration. However, larger groups also expose a greater number of

[2]Many more examples of social and economic networks can be found in Jackson (2008) and Newman (2010).

[3]There are exceptions but to address them we need a more specialized model. For example, as Chai (1993) suggests while it is true that exposure to the risk of infiltration keeps the size of resistance groups small, these groups can be further protected by having a hierachical structure where every person is in contact with no more that three other members (one above and two below).

participants to shocks in their network (new virus, an agent caught by the authorities, a macroeconomic downturn). How big the network will be it will depend on balancing the costs and benefits of participation. The risk of losing a member in clandestine operations might be unacceptable and thus would keep the size of the network small. In contrast, the international trade network has been expanding both by enlisting more trading partners and reducing barriers to trade.

Financial networks and systemic risk This is our first example of a directed network. In the description we offered above shocks are transmitted around the network from borrowers to lenders. The inability of a borrower institution to meet its obligations with its lenders can cause a cascade of failures through the system. The lenders of the initially failing institution might be unable to meet their own obligations and this process can keep going till the system is cleared (e.g., Eisenberg and Noe, 2001).[4] Shocks can also be transmitted from lenders to borrowers when the former group suddenly interrupts established credit lines that it has earlier provided to the latter group. Episodes of market freezes usually take place before the onset of a crisis, as lenders anticipate that borrowers will, in the near future, have a hard time repaying their debts (e.g. Diamond and Rajan; 2011).[5]

During a crisis, systemic losses can get magnified because of 'fire sales'.[6] As Shleifer and Vishny (1992) have shown the simultaneous liquidation of assets by multiple institutions can depress the market prices of these assets which in turn further deteriorates the balance sheets of other institutions holding similar assets thus potentially leading to more failures.

Firm linkages and macroeconomic fat tails Directed networks are also useful for understanding the causes of fat tails in the distribution of macroeconomic shocks. Recent work by Acemoglu *et al.* (2012, forthcoming-b) have shown that the interaction between the distribution of idiosyncratic shocks and the structure of the network can become a shock amplification mechanism that can explain 'abnormal' shocks at the aggregate level.[7] They analyze networks where nodes represent firms that buy from and sell goods to each other thus creating a web of complex relationships.

In Acemoglu *et al.* (forthcoming-b) they show that light-tailed risks (small deviations from the normal case) in conjunction with some lack of balance in terms of economic importance across the sectors of the economy can give rise to macroeconomic fat tails. One possible explanation for the deviation of the distribution of idiosyncratic shocks from the normal case can be related to what in the past macroeconomists have identified as aggregate demand externalities. Examples of such externalities have been offered in relation to search in labor markets (Diamond, 1982), coordination of economic activity (Cooper and John, 1988), market structure (Kiyotaki, 1988) and industrial development (Murphy *et al.,* 1989).

[4]For excellent literature reviews see Acemoglu, Ozdaglar and Tahbaz-Salehi (forthcoming-a), Babus and Allen (2009) and Glasserman and Young (2016).

[5]For a network approach to market freezes see Gabrieli and Georg (2014).

[6]See Shleifer and Vishny (2011) for a review of the literature.

[7]See also Carvalho (2014) for a lees technical exposition of this topic.

By introducing in our model aggregate external effects we were able to show that directed networks, such those considered above, though connected are not complete.

1.2. Related Literature

Our paper is related to a quickly expanding literature on the endogenous formation of economic and social networks. Early work focused on variants of the Jackson and Wolinsky (1996) connectionist model.[8] We focus our literature review on papers that consider network formation in environments with systemic risk.

As we mentioned above, the most closely related paper to ours is Blume *et al.* (2011). They restrict their attention to undirected graphs. However, the transmission of a shock is restricted to spread only through live nodes. In contrast, in the undirected graph version of our model all nodes are live but we allow for losses to be discounted with the distance of nodes from the one hit by the shock. The two models become identical by setting equal to one the probability of a node being live in Blume *et al.* (2011) and the discount factor in our model. In addition, to finding, as they do, that there exist stable networks that consist of disjoint fully connected subgraphs we also show that any stable network must have that structure.

Erol and Vohra (2014) also consider formation of undirected links and derive a similar result, however, from a network formation game that has a quite different structure. In their model any pair of agents linked together play a coordination game, each deciding whether to default or not and where their expected payoffs also depend on their beliefs about the default strategies of all other agents. At an earlier stage agents form undirected links anticipating the later stage possibilities.

Our work is also related to a number of papers in the finance literature that explore the links between network structure and systemic risk. There has been a lot of work in trying to understand which types of network structures are more vulnerable to systemic risk (e.g., Acemoglu *et al.*, 2015; Elliott *et al.*, 2014).

Recently, there has also been some work on the formation of such networks. Cohen-Cole *et al.* (2010) study competition in the financial market where participants form undirected links. In Babus (2013) financial institutions form links to insure themselves against the probability of system wide default. In the financial interpretation of our model financial institutions are participating in an interbank market. Acemoglu et al. (2014) also study network formation but they exogenously restrict the links that are allowed to be formed. When such restrictions are lifted then the complete network can be stable. We derive alternative equilibrium structures by introducing aggregate externalities.

Lastly, Caballero and Simpsek (2013) also consider externalities in financial markets within a network approach. They introduce the concept of ‘price complexity externality’ to refer to the negative externality imposed to the rest of the system by the liquidation of assets by failing institutions.

[8] See, for example, Bala and Goyal (2000), Dutta and Mutuswami (1997), Jackson and Watts (2002) and Watts (2002).

2. Undirected Networks

There are n agents represented as nodes on a graph (network). Let $N = \{1, ..., n\}$ denote the set of nodes. A link between two nodes indicates that the corresponding agents have a direct relationship. Let g^N denote the complete network where all agents are directly connected and let $G = \{g \mid g \subseteq g^N\}$ denote the set of all possible graphs. For two agents that are directly linked in network g we write $ij \in g$.

The following notations will be useful. We will write $g + ij$ for the network that we obtain when we add link ij to an existing network g. Similarly, we will write $g - ij$ for the network that we obtain after deleting link ij. A *walk* between agents i and j is defined as a sequence of agents beginning with i and ending with j such that for every pair of adjacent agents in the sequence there exists a direct link. A *path* between agents i and j is defined as a sequence of agents beginning with i and ending with j such that for every pair of adjacent agents there exists a direct link and each agent appears only once in the sequence (each node is distinct). We let $t(ij)$ denote the number of direct links in the shortest path between agents i and j. A *cycle* is formed by adding the link ij to a path between agents i and j. A *complete cycle* is a cycle where all the agents that belong on the cycle are connected with each other. An *empty cycle* is a cycle that there are no links between agents that belong on the loop and are not adjacent.

For any network g we let $C(g)$ denote the set of all distinct connected subgraphs and $D(g)$ the set of all isolated nodes. Then $g = (\cup_{g' \in C(g)} g') \cup D(g)$. We let g'^N denote that the subgraph g' is complete.

For any player i we define $T_k^i = \{j : t(ij) = k\}$, that is the set of agents with a shortest distance from agent i equal to k. Let $|T_k^i|$ denote the cardinality of the set. Then we have $\sum_{k=1}^{n-1} |T_k^i| = n - 1$. Notice that $|T_1^i|$ is equal to the *degree* of node i.

Next, we define the benefits and costs from network participation. With probability θ one of the nodes of the network is hit by a shock. Conditional on the occurrence of the shock all nodes are hit by the shock with equal probability. Thus, the unconditional probability that a node is hit by a shock is equal to $\frac{\theta}{n}$.[9] Agents derive benefit b from each direct link as long as one of the following two conditions holds: Either the network is not hit by a shock or the network is hit by the shock but they are not connected, neither directly nor indirectly, to the agent hit by the shock. There is no benefit from indirect links. The cost to an agent of being hit by a shock is equal to $c < 1$. Other agents of the network will suffer losses only if they are connected to the agent who is hit by the shock (they have to belong to the same connected graph). Say agent j is hit by a shock. For any agent i connected to agent j this indirect cost is given by $\delta^{t(ij)}c$, where $0 < \delta < 1$; thus, the cost is declining with the number of links in the shortest path between the two agents.

Suppose that $g' \in C(g)$ and let $|g'|$ denote the cardinality of g'. Then, the expected

[9] In Section 4, we discuss alternative specifications of the distribution of shocks.

net payoff $v_i(g', N)$ of agent $i \in g'$ is given by:

$$\begin{aligned} v_i(g', n) &= \left(1 - \frac{|g'|\,\theta}{n}\right) \left|T_1^i\right| b - \frac{\theta}{n} c \left(1 + \delta \left|T_1^i\right| + ... + \delta^{n-1} \left|T_{n-1}^i\right|\right) \\ &= \left(1 - \frac{|g'|\,\theta}{n}\right) \left|T_1^i\right| b - \frac{\theta}{n} c \left(1 + \sum_{k=1}^{n-1} \delta^k \left|T_k^i\right|\right) \end{aligned}$$

The first term is equal to the expected benefit derived from belonging in a subset of the network g represented by a connected subgraph g'. There subgraph has $|g'|$ nodes and thus the probability that one of these agents is hit by a shock is equal to $\frac{|g'|\theta}{n}$. As long as the no agent that belongs to the subgraph is hit by the shock agent i will derive a benefit b from each of the direct links where the total number of these links is equal to $|T_1^i|$. The second term is equal to the corresponding expected costs. Each agent fails with probability $\frac{\theta}{n}$. When an agent who belongs to the subgraph is hit by the shock all agents suffer a loss that is equal to c times a discount factor that depends on the shortest distance of the agent from the one hit by the shock.

2.1. Stability

As in Jackson and Wolinsky (1996) we use the notion of *pair-wise stability* to allow us to make predictions about the types of networks that are likely to form.

Definition 1 *A network, g, is stable if no agent i prefers to severe a link, and no pair of agents i and j prefers to form link ij.*

Thus, the formation of a new link requires the approval of both agents forming the link. But any players can severe a link unilaterally. This is a relatively weak notion of stability given that, as we will see below, allows for stable networks where every participant would prefer to severe all links simultaneously. In Section 4, we discuss alternative notions of stability where such cases would be eliminated. Still, the notion of pair-wise stability by sufficiently restricting the set of stable networks allows us to make interesting predictions. Moreover, as Jackson and Wolinsky (1996) note pairwise stability is independent of any particular dynamic process through which the network is formed.

Proposition 1 *(a) if $\left(1 - \frac{2\theta}{n}\right) b < \frac{\theta}{n}\delta c$ then the empty network is stable, and*
(b) if $(1-\theta)\, b > \frac{\theta}{n}\delta\,(1-\delta)\, c$ then the complete network, g^N, is stable.

Proof

(a) Consider the empty network and any pair of agents i and j. The probability that one of the two agents is hit by a shock is equal to $\frac{\theta}{n}$ in which case, if the link has been formed, the agent not hit by the shock will bear an indirect loss δc. With probability $1 - \frac{2\theta}{n}$, none of the two agents is hit by the shock in which case each receives benefit b. Thus, if $\left(1 - \frac{2\theta}{n}\right) b < \frac{\theta}{n}\delta c$ the two agents will decide not no form the link and the proof follows from the arbitrary choice of the pair.

(b) The payoff of agent i who is part of the complete network is given by:

$$v_i(g^N, n) = (1-\theta)(n-1)b - \frac{\theta}{n}c(1+(n-1)\delta) \tag{1}$$

Agent i by severing a link, say with agent j, loses the benefit b when there is no shock on the network that is with probability $1-\theta$. Now $t(ij) = 2$ and thus by severing the link the expected cost of participating in the network for agent i has been reduced by $\frac{\theta}{n}\delta(1-\delta)c$. The proof follows form comparing benefits and losses and the arbitrary choice of agents. □

Corollary 1 *If*

$$(1-\theta)b < \frac{\theta}{n}\delta(1-\delta)c < \frac{\theta}{n}\delta c < \left(1-\frac{2\theta}{n}\right)b \tag{2}$$

then neither the complete network nor the empty network are stable.

Our next result shows that stable networks always exist. In particular, as in Blume *et al.* (2011), we will show that we can always construct stable networks that consist of disjoint connected subgraphs. Then we will also show that any stable network has that structure.

Proposition 2 *Stable networks always exist.*

Proof We will prove the proposition in two steps. We will first show that the existence of a stable complete subgraph is sufficient for the existence of at least one stable network. Then we will show that a stable complete subgraph exists.

Lemma 1 *If there exists a stable complete subgraph g'^N then there also exists at least one stable network.*

Proof Let $\left|g'^N\right| = m$. Suppose that $n \bmod m = v$ and consider the network with $\frac{n-v}{m}$ complete subgraphs each with m nodes and 1 complete subgraph of size v. To prove the proposition we need to show that the complete subgraph of size v is stable. The expected payoff of an agent i belonging to one of the complete subgraphs of size m is given by:

$$v_i(g'^N, m, n) = \left(1-\frac{m\theta}{n}\right)(m-1)b - \frac{\theta}{n}c(1+(m-1)\delta) \tag{3}$$

One of the necessary conditions for the stability of the subgraph is that agent i does not want to severe a link. The new payoff of an agent who severs a link is given by $\left(1-\frac{m\theta}{n}\right)(m-2)b - \frac{\theta}{n}\left(c+(m-2)c^2+c^3\right)$, and thus the stability condition is given by:

$$\left(1-\frac{m\theta}{n}\right)b - \frac{\theta}{n}\delta(1-\delta)c > 0 \tag{4}$$

Next, we consider the stability of a complete subgraph of size v. The stability of the rest of the subgraphs implies that none of the agents belonging to the other subgraphs are willing to link with any agent belonging to another subgraph. Given that the left-hand side of (4) is decreasing in m and given that $m > v$ none of the agents in the complete subgraph of size v prefers to severe a link. □

Suppose that (2) holds; that is neither the empty network nor the complete network are stable. Clearly, if this is not the case existence of stable networks is trivially satisfied. Then the lemma implies that it is sufficient to show that a stable complete subgraph exists. The existence of a stable complete subgraph of cardinality m requires that two conditions are satisfied: (a) No agent prefers to break a link, that is $\left(1 - \frac{m\theta}{n}\right) b > \frac{\theta}{n}\delta\left(1-\delta\right)c$, and (b) that no isolated agent would like to join the graph, that is $\left(1 - \frac{(m+1)\theta}{n}\right) b < \frac{\theta}{n}c\left(\delta + (m-1)\delta^2\right)$. (Stability also requires that none of the agents belonging on the subgraph would like to join agents outside the graph but this constraint does not bind. Further, if no isolated agent would like to join the complete subgraph then this will be the case for any other agent belonging to any type of subgraph as joining an even larger network always decreases expected payoff.) Then it suffices to show that if (2) holds then there exists $m \in [2, n-1]$ such that the following inequalities are satisfied:

$$\left(1 - \frac{(m+1)\,\theta}{n}\right) b < \frac{\theta}{n}\delta c < \left(1 - \frac{m\theta}{n}\right) b \tag{5}$$

The proof follows from the observations that for $m = 2$ the second inequality is satisfied by (2) and for $m = n - 1$ the first inequality is satisfied by (2). □

The networks identified by proposition 2 are such that every $g' \in C(g)$ is complete.

Example 1 *Let $n = 9$, $\delta = 0.9$, $b = 5$, and $c = 1$. Then the network comprised of two complete subgraphs of sizes 5 and 4, respectively, is stable (see Figure 2.1). Notice that (4) is satisfied for these parameter values. In addition, we need to ensure that no agent from the size 4 subgraph would like to link with an agent from the size 5 subgraph. The expected payoff from creating the new link is given by*

$$(1-\theta)4b - \frac{\theta}{9}c\left(1 + 4\delta + 4\delta^2\right)$$

which is less than the expected payoff from not creating the link given by (3) when we use the values $n = 9$ and $m = 4$.

(5) is a sufficient but by no mean necessary condition. Even if a network might not be stable in the presence of isolated agents might be so in their absence. For example, a subgraph of size, say m' might not be stable if there are isolated agents it might be so if the smallest size complete connected subgraph in the network exceeds some minimum

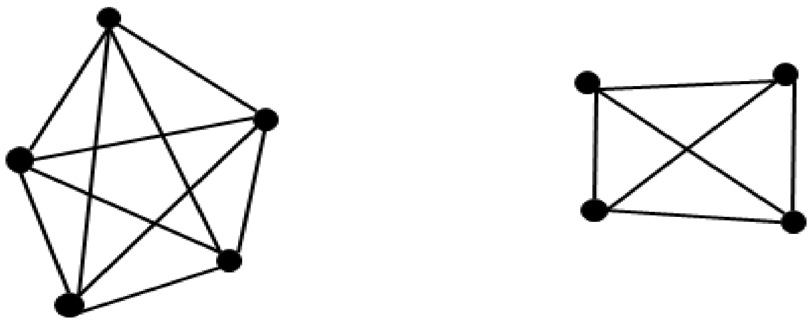


Figure 2.1: A Stable Network ($n = 9$, $\delta = 0.9$, $\theta = 1/3$, $b = 5$, $c = 1$)

value. An implication of the last comment is that a multiplicity of stable networks might exist.

Next, we extend the characterization of the stable networks described by the above result by providing an upper bound on the size of subgraphs that can be parts of stable networks. Define $m^{**}(n)$ as the maximum value of m such that the following inequalities hold $\left(1 - \frac{(m+1)\theta}{n}\right) b \geqslant \frac{\theta}{n} c \left(\delta + (m-1)\delta^2\right)$. m^{**} is equal to the largest size of fully connected subgraph that an isolated agent would wish to join and thus sets an upper bound for the size of such subgraphs. The lower bound can be an isolated agent. To see this suppose that $m^{**} = n - 1$. In that case, one stable network consists of a completely connected subgraph of size $n - 1$ and an isolated agent.

Next, we show that any stable network consists of disjoint fully connected subgraphs.

Proposition 3 *Any incomplete connected subgraph is not stable.*

Proof We prove the following results:

> **Lemma 2** *If a complete subgraph of size m is not stable because agents would prefer to severe a link then any connected subgraph of size m is also not stable because agents would prefer to severe a link.*
>
> **Proof** We need to consider two cases:
>
> (a) A broken link does to alter the size of the subgraph: Form (4) we know that instability implies that $\left(1 - \frac{m\theta}{n}\right) b < \frac{\theta}{n}\delta\left(1-\delta\right) c$. The left hand side shows the expected loss from breaking a link which does not depend on the structure of the subgraph as long as its size is equal to m. When the original graph is not

complete then it must be the case that for no agent the expected benefit of breaking a link can be lower than $\frac{\theta}{n}\delta(1-\delta)c$ (the expected benefit corresponding to breaking a link of a complete subgraph) and there must be at least one agent whose expected benefit must be higher. Expected costs once more decline by $\frac{\theta}{n}\delta c$ because of the direct link but now they can decline even more because the shortest path to other agents has increased. Further, after the break of a link in the complete network the shortest path between the two corresponding agents has increased to 2 with corresponding cost $\delta^2 c$ and when the subgraph is not complete the shortest path can be even higher.

(b) A broken link decreases the size of the subgraph: The lowest expected payoff that an agent can gain be severing a link is when the other agent is not connected to anyone else. This is because (a) the probability of being affected by a shock depends on the size of the subgraph which in this case only declines by 1, and (b) there are no additional benefits from cost reduction given that the shortest path to other agents is not affected. For such an agent the expected benefit from participating in the network prior to the break of the link would be is to $\left(1-\frac{m\theta}{n}\right)(m-1)b$ and the corresponding benefit after the break is equal to $\left(1-\frac{(m-1)\theta}{n}\right)(m-2)b$. There is also a reduction in expected costs by $\frac{\theta}{n}\delta c$. These conditions imply that the agent would prefer to break the link if $\left(1-\frac{m\theta}{n}\right)b-\frac{\theta}{n}(m-2)b-\frac{\theta}{n}\delta c<0$ which is implied by the instability condition of the complete subgraph. □

Lemma 3 *Consider a complete subgraph of size m where no agent prefers to severe a link. Then any incomplete connected subgraph of size m is not stable.*

Proof Notice that the fact that no agent prefers to severe a link when the subgraph is complete implies that (4) holds. Consider any incomplete connected subgraph of size m where no agent prefers to hold a link. (If this is not the case the lemma holds.) Then there exist agents i, j, k such that $ik \in g'$, $kj \in g'$, $ij \notin g'$ and both i and j would like to link with each other. We focus on the decision of agent i given that the decision of agent j is symmetric. We need to consider three cases:

(a) j is a terminal node: agent i's expected payoff from creating a link with agent j is given by the left-hand side of (4) and therefore is positive.

(b) Breaking link jk would divide the original subgraph into two distinct connected subgraphs: The expected payoff to agent i from creating link ij is higher than the expected payoff to agent k from maintaining kj. This is because agent k by breaking kj could benefit from (a) reducing the costs due to all indirect links through agent j, and (b) reducing the size of the subgraph and thus the likelihood of being affected by a shock. In contrast, agent i's decision does not affect the size of the subgraph. Moreover by creating the link ij the only additional cost is that the shortest path to all indirect links is reduced by 1. (See, Figure 2.2)

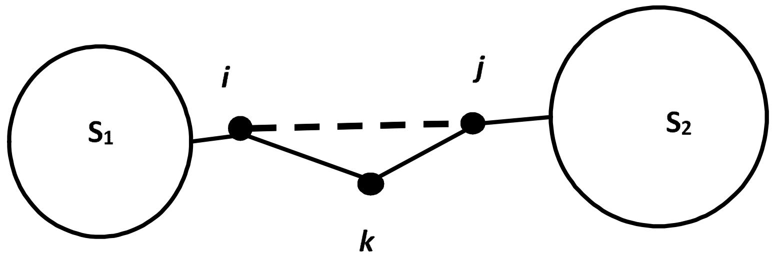


Figure 2.2: Proof of Lemma 3 (b)

(c) In the original subgraph there exists at least one cycle such that the subgraph remains connected after link jk is broken: Clearly jk belongs to the cycle. Agent i's net expected payoff from linking with agent j only depends on the distance of those agents from whom the shortest path includes link jk. (The shortest paths from all other agents is not affected by creating the link ij.) We need to consider two cases:

(i) ik belongs to all cycles: Consider all agents from whom the shortest paths to i after linking with j are through the link ij. Then the shortest paths of these agents to agent k are through link kj. (The reverse in not true as there are agents whose shortest paths from agent i are not through agent j - other way around the cycles - but their shortest path from agent k is through agent j.) From the above it follows that if agent k prefers to maintain link kj then agent i will prefer to create link ij. (See, Figure 2.3)

(ii) ik does not belong to any of the cycles: This implies that the links ik and jk are on opposite sides of any cycle. If an empty cycle does not exist then the arguments used in part (i) still hold. Suppose that there exists at least one empty cycle. If the number of links in this empty cycle is odd then the arguments used in part (i) above still hold. If the number of links in the cycle is even then there exists an agent l such that there exist two (shortest) paths from agent k to agent l that are equal. In this case, the shortest path form k to l is not affected by the decision of k to break or to retain link jk. However, for if link ij is created then the shortest path from agent i to agent l decreases

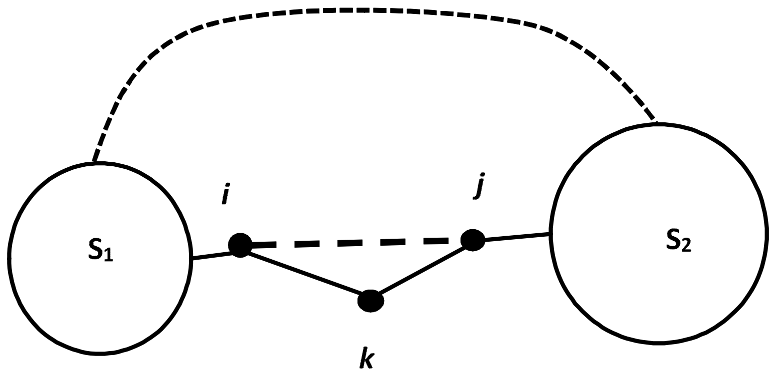


Figure 2.3: Proof of Lemma 3 (ci)

by 1. Thus, in such case there is a possibility that while agent k prefers to retain link, agent i might prefer not create link ij. However, given that the cycle is empty consider any three agents i', k', j' such that $i'k'$ and $j'k'$ belong to the cycle. By supposition these agents prefer not to break these links. Then, following the arguments so far, it must be the case that either agent i' prefers to create link $i'j'$ or there is an empty cycle where the links $i'k'$ and $j'k'$ are on opposite sides of the new cycle. The proof of the lemma is completed with the observation that the size of the connected graph is finite. (See, Figure 2.4) □

The proof of the Proposition follows from the above two results. □

Corollary 2 *Stable networks exist and each stable network consists of a collection of connected subgraphs.*

Stable networks respond to a trade-off between large size structures that bring more benefits to participants and small size structures that protect them from shocks. Given that indirect connections do not confer any benefits but are still a potential source they are absent in stable networks. The small size of hunter-gatherer societies might have indeed protected them against epidemics as the small size of resistance groups protects them against infiltration. In contrast, despite the losses in welfare resulting from the transmission of macroeconomic shocks there is still a tendency for expanding globalization by opening international borders to allow the movement of goods and services, inputs in production and financial capital.

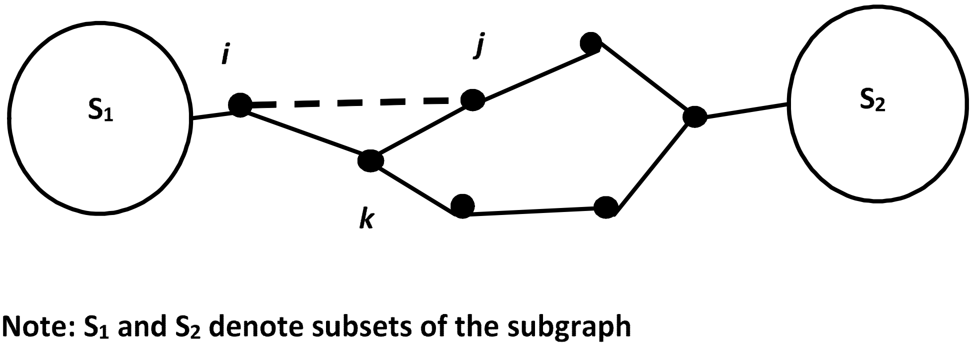


Figure 2.4: Proof of Lemma 3 (cii)

2.2. *Ex Ante* Efficiency vs *Ex Post* Systemic Losses

Up to this point we have been focusing on the types of network structures that are more likely to form. Now we turn our attention to relative performance. For the types of networks that we are interested there are two potentially interesting ways of measuring performance. The first one is having a measure of *ex ante* performance as in Jackson and Wolinsky (1996) who used the sum of expected utilities of network participants. The second way to measure performance is related to the very nature of these networks where the potential systemic losses after a shock can be significant. Clearly, these losses are minimized by not having any links, that is all agents are isolated, $|D(g)| = n$. Instead, we are going to identify among all stable networks the one that minimizes *ex post* losses.

Definition 2 *A network g^* is efficient if it maximizes the sum of the payoffs of all agents:*[10]

$$g^* = \arg\max_g \sum_{i=1}^n v_i(g,n) = \arg\max_g \sum_{g' \in C(g)} \sum_{i \in g'} v_i(g',n) - |D(g)| \frac{\theta}{n} c$$

Proposition 4 *Efficient networks are such that every $g' \in C(g)$ is complete.*

Proof The payoff of an agent i belonging to one of the complete subgraphs of size m is given by (3). Subtracting $-\frac{\theta}{n}c$ and dividing by $m-1$ we get $\left(1 - \frac{m\theta}{n}\right) b - \frac{\theta}{n}\delta c$ which

[10]Notice that the expected payoff of each $i \in D(g)$ is equal to $-\frac{\theta}{n}c$. This expression cancels in all our derivations but the ones in the last section were we introduce aggregate externalities. For this reason we opted to keep it rather making an *ad hoc* introduction later in the paper.

is equal to each agent's net expected payoff from each node. We need to consider two cases:

(a) $\left(1 - \frac{m\theta}{n}\right) b - \frac{\theta}{n}\delta c \geq 0$: Next, consider any other connected subgraph of cardinality m that is not complete. From each directly linked node an agent gets exactly the same expected payoff as the one derived form being a member of a complete subgraph. The expected payoff derived form nodes not directly linked is negative due to connectivity. Thus the total payoff is maximized when the subgraph is complete.

(b) $\left(1 - \frac{m\theta}{n}\right) b - \frac{\theta}{n}\delta c < 0$: In this case net expected payoff received form each node in the complete graph is negative and clearly the payoffs of all agents would have been higher had they been isolated. □

Corollary 3 *The efficient network is stable.*

Proof As we argued above an agent cannot achieve a higher payoff by breaking a single link. It is also the case that cannot achieve a higher payoff by forming a link. Suppose that this is not the case and can achieve a higher payoff by connecting with an isolated agent. But the new payoff would be exactly the same as the one that the agent would obtain form being in a complete subgraph of size $m+1$. This would also be true for any other agent contradicting the assumption that the original network was efficient. □

By maximizing (3) with respect to m we can find the size of a completely connected subgraph $\hat{m}$ that offer the maximum expected payoff to it members. Given that $\hat{m}$ is probably not an integer we compare the payoffs of subgraphs of sizes equal to the first integer higher than $\hat{m}$ and the first integer lower than $\hat{m}$. From now on we ignore this complication. Setting the f.o.c. equal to 0 and solving for m we get:

$$\hat{m} = \left(1 + \frac{n}{\theta} - \frac{c\delta}{b}\right)/2$$

Notice that for any given size of the network, n, the higher the probability that the network will hit by a shock the lower the size of the optimal subgraph. If $n \bmod m = 0$ then clearly the network consisting of completely connected subgraphs of size $\hat{m}$ maximizes ex ante efficiency. If $n \bmod m > 0$ then not all agents will be receiving the same payoff and the most efficient network might not feature subgraph of size $\hat{m}$. However, for high values θ, c and δ or low values of b, $\hat{m}$ will be small relatively to n in which case it is more likely that the efficient network mainly consists of subgraphs of size $\hat{m}$.

Next, we consider the relationship between stability and systemic losses. Clearly, the smaller the size of subgraphs is the lower will be the size of systemic losses following a shock. Therefore, we are looking for the smallest fully connected subgraph that is stable when all other subgraphs have the same size.[11] Denote the size of such subgraph by $\check{m}$.

[11] Once more, we need to do a bit more work when n is not divisible by that particular size.

An agent belonging to a fully connected subgraph of size $\check{m}$ would prefer not to link with another agent belonging to another similar graph if the following inequality holds:

$$\left(1-\frac{m\theta}{n}\right)(m-1)b-\frac{\theta}{n}c\left(1+(m-1)\delta\right)>\left(1-\frac{2m\theta}{n}\right)mb-\frac{\theta}{n}c\left(1+m\delta+(m-1)\delta^{2}\right)$$

The left-hand side is the expected payoff from being part of a completely connected subgraph of size m as given by (3). If the agent links with an agent in another similar subgraph the size of the new network will double and thus the probability that the subgraph will be hit by a shock doubles. There is the additional benefit b from the extra link but also there are additional costs. There is an additional expected cost $\frac{\theta}{n}c\delta$ related to the new link and an additional expected cost $\frac{\theta}{n}c(m-1)\delta^2$ from the new indirect links. $\check{m}$ is the smallest value of m such that the above inequality is satisfied. In that case a network consisting of fully connected subgraphs of size $\check{m}$ is stable.

Example 2 *Let $n=6$, $\delta=1$, $b=1$, $c=7$, and $\theta=\frac{1}{2}$. Then $\hat{m}=3$. The above inequality is satisfied for $m=2$ but not for $m=1$ and thus we have $\check{m}=2$. Notice that this example also trivially satisfies the stability condition for the complete network.*

The above example identifies a tension between stability, efficiency and the size of systemic losses. It is not surprising that if to minimize systemic losses we need very low connectivity as in this case we have completely ignored the benefits from having the network. The observation that stability can be satisfied with networks that are much larger than those that maximize efficiency, as we discuss in more detail in section 4, is not necessarily due to the notion of stability that we use. When agents make decisions about forming or breaking a link they ignore the negative impact that these decisions have on the payoffs of other agents.[12]

3. Directed Networks

Up to this point we have treated symmetrically two agents forming a link. In our general model the paths formed by the links of the network capture the contagion flows following a shock. Thus far we have allowed the flows to follow both directions defined by a link. After a link is formed when any one of the two agents is hit by a shock then the other agent also suffers losses. However, in many applications contagion flows only in one direction which depends on the nature of the relationship between the two agents. For example, for any two linked banks in banking networks there is a lender bank and a borrower bank. When the borrower bank is hit by a shock and is unable to meet its obligations to the lender bank the latter also suffers a loss. The lender bank might also play the role of a borrower bank in another link in which case the shock can be further transmitted.

We will use directed links to capture these one way flows. In what follows ij captures not only the fact that agents i and j are linked but also that shocks are transmitted from agent i to agent j. Graphically, there will be an arrow between nodes i and j pointing at

[12] As Acemoglu *et al.* (2014) show in financial markets ignoring this exteral effect leads to overlending.

node j. Using the new interpretation of ij we can define, *directed walks*, *directed paths* and *directed cycles* using the definitions for walks, paths and cycles offered in the last section. $t(ij)$ now denotes the shortest directed path from i to j,[13] $T(in)_k^i = \{j : t(ji) = k\}$ and we let denote the set of agents with a shortest distance to agent i equal to k and $T(out)_k^i = \{j : t(ij) = k\}$ denote the set of agents with a shortest distance from agent i equal to k. Notice that the cardinality of these sets for $k = 1$ are the *in-degree* and the *out-degree* of node i, respectively. For all agents such that there are no paths leading to them from agent i we set $t(ji) \approx \infty$. Lastly, a *Hamiltonian path* is a directed path that visits each node exactly once while a *Hamiltonian cycle* is a Hamiltonian path with an additional link from the last node to the original node.

Next, we define the benefits and costs from participation in directed networks. As above, with probability θ one of the agents of the network is hit by a shock. Conditional on the occurrence of the shock all agents are hit by the shock with equal probability. Thus, the unconditional probability that an agent is hit by a shock is equal to $\frac{\theta}{n}$. As long as there is no agent hit by a shock each agent obtains a benefit b form each direct link in either direction. As above, an agent hit by a shock suffers cost c. Other agents of the network will suffer losses only if they are connected to the agent who is hit by the shock by a directed path. Say agent j is hit by a shock. For any agent i connected to agent j this indirect cost is given by $\delta^{t(ji)}c$, where $0 < \delta < 1$; thus, as above the cost is declining with the number of links in the shortest path between the two agents, however, now we restrict our attention to directed paths beginning from agent j who is hit by the shock and reaching other agents. The loss of benefits that each agent suffers following any shock it will depend on the connectedness of the network. When agent j is hit by a shock all links located on directed paths beginning with agent j are affected and the corresponding benefits are lost to both agents of each link. Then consider the network $g'(j)$ obtained from the original network g after we have eliminated all affected links. Then each agent will keep the benefits from all their remaining direct links in either direction. Then we can write the expected payoff function of agent i from participating in the original network g as:

$$v_i(g,n) = \left(1 - \frac{|g|\,\theta}{n}\right)|T_1^i|\,b - \frac{\theta}{n}\left(c - |T_1^i|_{g'(i)}\,b + \textstyle\sum_{j(\neq i)\in g'(i)}\left(\delta^{t(ji)}c - |T_1^i|_{g'(j)}\,b\right)\right)$$

where $|T_1^i| = |T(in)_1^i| + |T(out)_1^i|$ and $|T_1^i|_{g'(j)}$ is defined in a similar way for the network obtained after agent j is hit by a shock. The costs are calculated as in the case for undirected networks but know only those paths leading to i are included. Notice that for all agents such that there is no directed path leading from them to agent i, $\delta^{t(ji)} \approx 0$.

The following example describes the expected net payoffs obtained from being part of a complete directed networks of size 3. Understanding these simple networks is crucial for the general analysis of directed networks of any size greater than 3.

Example 3 *Suppose that $n = 3$ (i, j, k). There are two possible types of complete directed networks (see, Figure 3.1)*

[13]Keep in mind that $t(ji) \neq t(ij)$.

Figure 3.1: (a) $n = 3$ Cycle; (b) $n = 3$ Complete Order

(a) Cycle: The links are ij, jk, ki. The network is symmetric as all agents have exactly the same net expected payoff given by:

$$v(g,3) = (1-\theta)\,2b - \frac{\theta}{3}\left(c + \delta c + \delta^2 c\right)$$

In this case when any agent is hit by a shock all benefits are lost any all agents will suffer losses.

(b) Complete Order:[14] *The links are ij, ik, jk. The net expected payoffs are given by:*

$$v_i(g,3) = (1-\theta)\,2b - \frac{\theta}{3}(c - 4b)$$

$$v_j(g,3) = (1-\theta)\,2b - \frac{\theta}{3}\,(c + \delta c - 3b)$$

$$v_k(g,3) = (1-\theta)\,2b - \frac{\theta}{3}\,(c + 2\delta c - 3b)$$

When agent i is hit by a shock all links are affected, when agent j is hit by a shock only link jk is affected and when when agent k is hit by a shock none of the links are affected. Remember that agents keep receiving benefits from links that are not affected.

The above example illustrates how small changes in connectivity can have large aggregate and distributional effects.

[14] A complete directed graph of size n is a *complete order* if we can label the nodes $v_1, ..., v_n$ such that there is a link form v_i to v_j, link ij, if and only if $j < i$. Noptice there are links form v_n to all other nodes and there are links to v_1 from all other nodes. A complete directed graph is also known as a *tournament.*

3.1. Stability

Definition 3 *A network, g, is stable if no agent i prefers to severe a link, and no pair of agents i and j prefer to form either link ij or link ji.*

The definition of stability is similar as that used in the case of undirected networks. The only difference is that now it requires that any pair of agents not linked do not want to form any of the two types of directed links.

Lemma 4 *Suppose that $(1-\theta)\,b+\frac{\theta(n-1)}{n}b>\frac{\theta}{n}\delta c$. Then the empty network is not stable.*

Proof Consider two isolated agents i and j. It suffices to consider the expected payoff of creating the link ij for agent j who suffers losses when any of the two agents is hit by a shock. The first term of the left-hand side of the inequality shows agent j's expected payoff from creating the link in the absence of any shock. Agent j will lose the benefit of the link only when agent i is hit by a shock and therefore even when there is a shock with probability $\frac{\theta(n-1)}{n}$ the link is intact and obtains benefit b. The right hand side shows the net expected cost from creating the link conditional on one of the two agents is hit by a shock. Keep in mind that an isolated agent is also hit by a shock with probability $\frac{\theta}{n}$ and suffers a loss c. After the creation of the link this cost is still there, however, with probability $\frac{\theta}{n}$ agent i is hit by the shock and the expected loss to agent j equal to $\frac{\theta}{n}\delta c$. Thus, when the inequality holds both agents prefer to create the link. □

When the empty network is not stable a link ij is always beneficial to agent i (the origin of the link). In contrast, whether the link is beneficial to agent j it will depend on the distribution of shortest paths that include link ij. If any agent along these paths is hit by a shock agent j will also suffer a loss. In contrast, the only benefit that agent j obtains from such paths is from the link to agent i. Below we will show that small changes in this trade-off can have large consequences for the structure of the network and, hence, for aggregate losses due to shocks. For making these comparisons we define an agent as *critical* if being hit by a shock implies that all links of the network are affected. We begin by looking at the extreme case where a path of two links pointing in the same direction is unstable.

Proposition 5 *Suppose that the empty network is not stable and $(1-\theta)\,b+\frac{\theta(n-2)}{n}b<\frac{\theta}{n}\delta\,(1+\delta)\,c$. Then the only stable network is a complete order tournament.*

Proof Consider three agents i,j,k and links ij and jk. The left-hand side of the inequality is equal to the expected payoff of agent k. Agent k will lose the benefit b from link jk when either agent i or agent j is hit by a shock (but not agent k). Thus, agent k will benefit from the link if either there is no shock (probability $1-\theta$) or there is a shock but it does not affect the link (probability $\frac{\theta(n-2)}{n}$). The right-hand side of the inequality is equal to the net expected cost to agent k from keeping the link who suffers losses when either agent i or agent j is hit by a shock.

Lemma 5 *A directed cycle of three agents is not stable.*

Proof The expected net payoff of any agent who is part of a directed cycle of three agents is equal to

$$(1-\theta)\,2b+\frac{\theta(n-3)}{n}2b-\frac{\theta}{n}\left(1+\delta\left(1+\delta\right)\right)c$$

Each agent benefits from two links as long as none of the three agents is hit by a shock. They suffer losses when any of the three agents is hit by a shock. Next, consider the benefit from breaking a link. Clearly, an agent who does that would break the incoming link. The expected payoff after the break of the link is given by

$$(1-\theta)\,b+\frac{\theta(n-1)}{n}b-\frac{\theta}{n}c$$

As long as the agent is not hit by a shock the link remains intact. Moreover, there is a loss only when the agent is hi by a shock. Then, for the agent to prefer to keep the link the following inequality must hold:

$$(1-\theta)\,b+\frac{\theta(n-5)}{n}b\geqslant\frac{\theta}{n}\delta\left(1+\delta\right)c \tag{6}$$

which by supposition is false. □

Lemma 6 *All incomplete connected subgraphs are not stable.*

Proof Suppose that this is not the case. Consider an incomplete connected subgraph where there is no agent who prefers to severe a link. We will show that there exists at least one pair of agents that would like to form a link. From the above lemma we know that the only way to fully connect three agents together is by a complete order. We also know that we cannot have links ij and jk without link ik (if this is not the case then agent k would prefer to severe link jk). To complete the proof we need to demonstrate that any group of three agents is fully connected by a complete order. We need to examine two cases:

(a) Consider three agents i, j, k and links ij and kj. Thus, both links are directed to agent j. Without any loss of generality, consider the creation of link ij. Agent j should agree to form the link unless there is another agent l and link li, but not link lj, in which case agent j will end up at the end of the path created by links li and ij. (notice that given that, by supposition, k did not prefer to severe the link ik this also implies the existence of link lk). But then consider the formation of the link lj. The only reason that agent j would prefer not to form this link is because there is another agent m and link ml. Given that the subgraph has a finite size we conclude that there is always a link that agent j would like to form and given that the proposed links are directed to agent j we have a contradiction.

(b) Consider three agents i, j, k and links ij and ik. Thus, both links are directed from agent i. Without any loss of generality, consider the creation of link jk. Agent k should agree to form the link unless there is another agent l and link lj, but not link lk, in which case agent k will end up at the end of the path created by links lj and jk. But then consider the formation of the link lk. The only reason that agent k would prefer not to form this link is because there is another agent m and link ml. Given that the subgraph has a finite size we conclude that there is always a link that agent k would like to form and given that the proposed links are directed to agent k we have a contradiction.

The above results imply that any group of three agents in the subgraph form a complete order and thus the subgraph must be complete. □

Harary and Moser (1966) have shown that any complete graph that does not have a three-agent cycle is a complete order. This implies that there is one agent who is critical for the subgraph. In order to complete the proof we are going to show that any complete subgraph of size less than n is not stable. There are two cases to consider:

(a) Isolated agents: Suppose that there exists an isolated agent and a complete order subgraph. A link directed from the isolated agent to the critical agent would increase the expected payoff of both agents.

(b) A disconnected group of complete order subgraphs: A link, in any direction, between the two critical agents would increase the expected payoff of both agents.

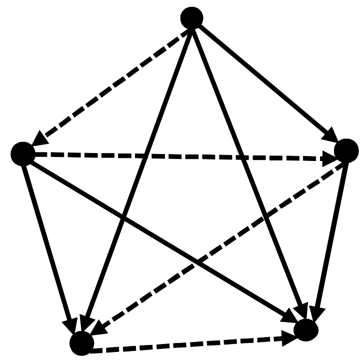
Thus, the only stable network is a complete order tournament. □

In the case of the above proposition there is a single critical agent associated with the single Hamiltonian path (see, Figure 3.2) For the above result, we have imposed the constraint that directed paths of length two are not stable. Next, we relax the constraint by allowing directed paths of length two but not directed paths of length three. The following result identifies conditions such that there exists a Hamiltonian cycle; that is every agent is critical.[15]

Proposition 6 *Suppose that (a) the empty network is not stable, (b)* $(1-\theta)\,b+\frac{\theta(n-2)}{n}b > \frac{\theta}{n}\delta\,(1+\delta)\,c$, *(c)* $(1-\theta)\,b+\frac{\theta(n-3)}{n}b < \frac{\theta}{n}\delta\,(1+\delta\,(1+\delta))\,c$, *and (d)* $c > \frac{2b}{\delta^2}$. *Then for odd values of* n *there exist stable tournaments where every agent is critical and for even values of* n *there exist stable tournaments where* $n-1$ *agents are critical.*

Proof Inequality (b) states that the net expected payoff of an agent at the end of a directed path of length two is positive. Inequality (c) states that the net expected

[15] There is quite a lot of work trying to establish the maximum number of hamiltonial paths and hamiltonian cycles in tournaments (e.g. Adler *et al.,* 2001). It is well known that the number can be very large. Here, we are interested in the existence of such paths and cycles when we impose restrictions on the maximum allwable shortes path.



Note: ……. Hamiltonian path

Figure 3.2: A Complete Order for $n = 5$

payoff of an agent at the end of a directed path of length three is negative. From Lemma 7 we know that inequality (b) is not sufficient for the existence of cycles of length three. Lemma 7 also identifies (6) as a necessary condition for such cycles. Then as long as inequalities (c) and (6) are jointly satisfied then cycles of length three are stable. This will be the case when inequality (d) holds. We will prove the proposition by construction:

(a) n is odd: Consider the complete directed network (tournament) where the in-degrees and the out-degrees of all nodes are equal to $\frac{n-1}{2}$. The adjacency matrix is given by

	1	2	3	...	$\frac{n-1}{2}$	$\frac{n-1}{2}+1$	$\frac{n-1}{2}+2$	...	$n-2$	$n-1$	n
1	0	1	1	...	1	1	0	...	0	0	0
2	0	0	1	...	1	1	1	...	0	0	0
3	0	0	0	...	1	1	1	...	0	0	0
...	...	...	...	...	...	...	...	...	...	...	...
$\frac{n-1}{2}$	0	0	0	...	0	1	1	...	1	1	0
$\frac{n-1}{2}+1$	0	0	0	...	0	0	1	...	1	1	1
$\frac{n-1}{2}+2$	1	0	0	...	0	0	0	...	1	1	1
...	...	...	...	...	...	...	...	...	...	...	...
$n-2$	1	1	1	...	0	0	0	...	0	1	1
$n-1$	1	1	1	...	0	0	0	...	0	0	1
n	1	1	1	...	1	0	0	...	0	0	0

Thus, if we arrange the agents around a circle, there are links from each agent to the next $\frac{n-1}{2}$ agents. There are also links directed to each agent from the previous $\frac{n-1}{2}$. Given that there exists a link directed from each agent to the next one moving clockwise there is a Hamiltonian cycle (there are many). We also need to show that the shortest path between any two agents i and j does not exceed two. Consider the shortest path from j to i. There are two possibilities: If the link is ji the shortest path is equal to one; if the link is ij then, by construction, there exists an agent l and links jl and li, so that the shortest path is equal to two.

(b) n is even: Consider any set of $n-1$ agents and construct a completed subgraph as above. Next, create links directed form each agent in the subgraph to the isolated agent.

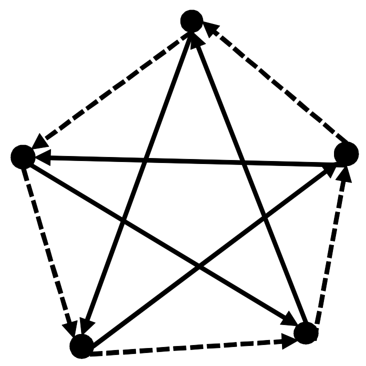
The only agent who is not critical is the agent who was isolated. This completes the proof. □

According to Proposition 5 when chains (shortest path greater than one) are too costly the only stable network is the complete order tournament where there is exactly one critical player. In Proposition 6 we have shown that when we allow for shortest paths equal to two then the conclusions dramatically change. There are stable tournaments where every agent is critical because there exists a Hamiltonian cycle (see, Figure 3.3). Any shock will affect all agents. Of course, the complete order tournament is still stable as there are many other tournaments with the number of critical agents ranging from one to n. In fact, it might be possible to construct stable networks that are not connected but which are comprised of sets of disjoint complete subgraphs. The reasoning behind this argument is based on what we know from our results related to undirected networks. However, there is a crucial difference. When the networks are directed the only network with isolated agents that can be stable is the empty network. As long as the empty network is not stable then any agent belonging to a connected subgraph would prefer to link with an isolated agent (in either direction but an outgoing link would be preferable) and any isolated player would definitely prefer to link with a connected subgraph as long as the link is outgoing (the agent might also prefer an incoming link).

The last observation suggests that the formation of stable networks that are not connected are less likely. Furthermore, our results for undirected networks also suggest that the only stable networks are complete. We have already shown that this is the case when the shortest path cannot exceed one. It follows that relaxing this constraint should not alter our conclusion that incomplete networks are not stable. However, the real-life directed economic networks that have the general structure of our model (e.g. input-output and financial) are connected but incomplete. Below we consider a simple extension of our basic model that will restrict the connectivity of stable structures.

3.2. Aggregate Externalities

Up to this point, we have assumed that the cost c is independent of the number of agents that are affected by the shock. However, both the macroeconomics and the financial economics literatures suggest that there exist mechanisms generating aggregate externalities



Note: ……. A Hamiltonian cycle

Figure 3.3: Hamiltonian Cycle for $n = 5$

that exacerbate the impact of shocks on each market participant. We capture these externalities by allowing the cost associated with a shock to be increasing in the number of affected agents, $\hat{n}$. Thus, we now write $c(\hat{n})$, where $c(1) > 0$ and $c' > 0$. It is clear that, other things equal, the likelihood that the complete network is stable declines with the size of the network. Offering a complete characterization of stable networks is a very complex problem and beyond the scope of this work. Nevertheless, the following results identify some of the properties of stable networks.

Proposition 7 *Suppose that there exists a positive integer z such that*

$$\left(1 - \frac{\theta}{z}\right)(z-1)b - \frac{\theta}{z}c(z) < 0 \tag{7}$$

Then for $\delta \approx 1$ the expected net payoffs of all agents belonging to any connected network of size $n \geqslant z$ that is complete are negative.

Proof The left-hand side of the inequality is equal to the expected payoff of the central agent of a size z star network where all the links are outgoing form the central node. As long as the central agent is not hit by a shock the links remain intact and when the central agent is hit by the shock the cost depends on the size of the network. The expected cost of belonging to any network of size z that has a Hamiltonian path must be at least $\frac{\theta}{z}c(z)$. Completeness implies the existence of a Hamiltonian path which in turn implies that the expected cost of any critical agent is at least

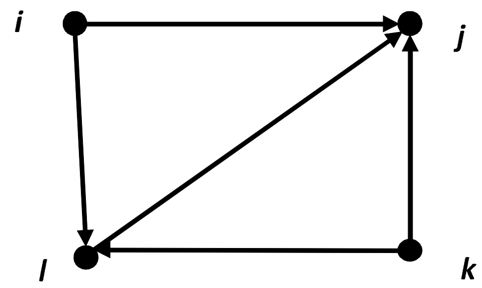


Figure 3.4: A connected but incomplete network, $n = 4$

$\frac{\theta}{z}c(z)$ (much higher if there is a Hamiltonian cycle where everyone is critical). For all other agents the expected cost depends on the length of the shortest paths separating them from other agents. They also suffer losses when other agents are hit by a shock, however, unless the agent who fails is critical the number of agents affected will be less than z, and thus $c(z)$ will be lower. However, for relatively high values of δ the additional costs will exceed the gain. Then, the expected benefit of belonging to any network of size z that has a Hamiltonian path must be at most $\left(1 - \frac{\theta}{z}\right)(z-1)b$ which corresponds to the case where the agent is linked to all other agents. □

The above result does not imply that these complete networks are not stable as breaking a link might not necessarily increase the expected payoff. However, as we explain in Section 4, this will not be the case if we introduce a stronger notion of stability. Moreover, any network with isolated agents cannot be stable. This is because creating a link directed from the isolated agent to any agent in the network will increase the payoff of both agents. Thus, stable networks are very likely to be connected but incomplete (see, Figure 3.4). In this example, if $c(4)$ is sufficiently high then agents i and k will prefer not to link in either direction as in that case a failure of the agent from where the link originated would affect the whole network thus increasing the cost to $c(4)$. As the network stands the maximum number of agents that will be affected after a shock is 3 in which case the cost will be equal to $c(3)$.

The general effect of aggregate externalities is to decrease the average number of agents that can be affected by a shock. However, that number can be small even in networks that are connected but not completely so.

4. Final Remarks

In this section, we consider the implications for our main results of changing some of the main assumptions of our model.

Stability The notion of pairwise stability that we have used throughout this paper is weak. In some cases , it allows for stable networks that would not survive a stronger notion while in other cases has complicated proofs. Below we discuss a couple of alternative notions of stability one related to the break of links and the other two the formation of new links.

Following Jackson and Wolinsky (1996) we have assumed that agents cannot break more than one link. Alternatively, consider the case where agents can break as many links as they like. One implication of this change is that the proof of part (cii) of Lemma 3 can be greatly simplified. While all other steps of the proof are straightforward, this particular case is not and only arises because our stability notion does not allow agent k to break both links on the cycle. Even if the expected payoff from retaining these two links is negative there is no benefit in severing only one of them. However, agent i by not accepting link ik avoids the losses. Under the alternative notion of stability k would break both of these links thus violating the supposition that the subgraph is stable thus leading to a contradiction.

The notion of pairwise instability might allow stable networks where each agent's next expected payoff is less that $-\frac{\theta}{n}c$. In contrast, such networks would not be stable had we allowed agents to break more than one link. We identify such a case above when we considered the impact of aggregate externalities on network formation. By breaking a single link an agents loses the benefit of having the link without reducing the cost significantly as the probability that the network is hit by a shock has remained the same.

Lastly, our notion of stability does not allow as deviations the simultaneous formation and breaking of links by groups of more than two agents. Consideration of such deviations would require the consideration of stronger equilibrium concepts such as coalition-proof Nash equilibrium.

Discounting We have introduced discounting (decay) in our model to capture the possibility that costs related to shocks are decreasing in the shortest distance from the agent hit by the shock. Further, to keep the exposition simple we have followed other examples in the literature (see, Jackson and Wolinsky, 1996; Watts, 2002) and have assumed geometric discounting. Our results still hold if we allow for a weakly decreasing decay function.

Distribution of shocks We have only allowed shocks that directly affect only one agent. Allowing for multiple shocks, either independent or correlated, would definitely affect quantitatively our results but not qualitatively. In all cases multiplicity of shocks would increase the parameter space within which the empty network is stable. For the case of undirected graphs it would also decrease the size of stable disjoint fully connected subgraphs. For the case of directed graphs without aggregate externalities, it would make more likely the formation of networks that are not fully connected while when aggregate externalities are present it would decrease the average number of agents that are affected by a shock.

Nodes and links In many interesting applications of directed graphs links van be bidirectional. For example, in financial networks two institutions can hold claims against each other. Generally, bankruptcy procedures do not allow the bilateral clearance of such claims after the failure of one institution which would violate priority rules (Eisenberg and Noe, 2001). Allowing for bidirectional links would not affect our analysis. We only observe that as the number of such links increases the network behaves more as an undirected one.

Further, in many applications of directed graphs (financial and macroeconomic networks) links and nodes can be weighted. Weights on links would capture the size of the transaction while weights on nodes would capture the size of the institution and thus potentially the probability of being hit by a shock.

Dynamics In the present work, we have concentrated on the properties of networks that in principle could be formed but we have ignored the dynamics of network formation and thus potentially the likelihood of these networks being formed. Our main objective has been to show that when we consider the formation of networks that transmit shocks we need to take into account the direction of the links. The main message of the paper does not depend on any particular dynamics. However, such dynamics can be important when we consider particular applications.[16] Moreover, we might be able to eliminate some of the less appealing stable networks (e.g. those where each agent's expected net payoff is negative) as dynamically unstable.

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