# On the Core of Auctions with Externalities: Stability and Fairness*

Seungwon (Eugene) Jeong†

April 6, 2017

**Abstract**

In auctions with externalities, it is well-known that the core can be empty, which is undesirable in terms of stability and "fairness." Nevertheless, some auction outcome should be chosen, and we show that the core is nonempty if payment refusals (often unrealistic in practice) are not allowed. In fact, we can categorize deviations into two: "pay more" and "refuse to pay." While "refuse to pay" may be unrealistic since normally bidders cannot refuse to pay, "pay more" is undesirable since there exist bidders who are willing to pay more than the final price, i.e., "justified envy." We show that there always exists an efficient outcome that might be unstable (by refusing to pay) but free of justified envy. The core and the Vickrey-Clarke-Groves (VCG) mechanism are closely related, e.g., any core payment of bidders is at least the VCG payment. While a loser's payment is necessary for core-selecting mechanisms, auctions with no loser's payment are widely used. Interestingly, when a loser's payment is not allowed, a core outcome no longer needs to be efficient, and no payment refusals ensure a nonempty core.

*Keywords*: core, auction, externalities, justified envy, core-selecting mechanism

*JEL Classification*: D44, C71, D62

## 1 Introduction

When Ukraine agreed to destroy nuclear weapons and signed the Nuclear Nonproliferation Treaty in 1994, Ukraine received high payments from both United States and Russia (Jehiel et al., 1996). That is, Ukraine received money without selling nuclear weapons, and US and

*Please check `http://tinyurl.com/CAEpaper` for the most up-to-date version.

†Department of Economics, University of Bristol, The Priory Road Complex, Bristol, BS8 1TU, UK. eugene.jeong@gmail.com. I am grateful to Francesco Giovannoni, Timo Hiller, Joosung Lee, In-Uck Park, and Siyang Xiong for helpful comments. All remaining errors are mine.

Russia paid even without "buying" nuclear weapons. As in the seminal paper on auctions with (identity-dependent) externalities (Jehiel et al., 1996), when externalities exist, a losing bidder's payoff may not be zero depending on the winner. Thus, each bidder has a different maximum willingness to pay in order to beat each competitor. Interestingly, losing bidders may also be willing to pay in order to avoid the worst allocation. These externalities are quite common in auctions where commercial bidders participate, e.g., in spectrum license auctions, the size of externalities can depend on the competitor's coverage map. Moreover, when externalities are real concerns, it is most likely a high-stakes auction.

As billions of dollars of spectrum licenses are sold by core-selecting package auctions (Ausubel and Milgrom, 2002; Milgrom, 2007; Day and Raghavan, 2007; Day and Milgrom, 2008; Erdil and Klemperer, 2010; Day and Cramton, 2012), the core is often used as stability and "fairness" concept in auctions.[1] For instance, if there are bidders who are willing to *pay more* than the final price, the seller and the bidders may deviate. Even if the deviation is not allowed for some reasons (e.g., by laws), the bidders may claim that this auction is "unfair," and the seller may also regret lower revenue.

In addition, externalities introduce some interesting (but may be less undesirable than the aforementioned *paying more* case) deviations by *refusing to pay* to the seller. First, if some losing bidders suffer too large negative externalities due to the winner, they rather "bribe" the winner not to buy, i.e., the winner refuses to pay. Second, losing bidders can be better off refusing to pay if the worst allocation without payments is better than the current allocation with payments. If an outcome is in the core, any kind of deviation is impossible.[2]

While the core property is desirable, Jehiel and Moldovanu (1996) show that the core can be empty in auctions with externalities: the core is nonempty only when the valuation of the winner is so high that all other valuations and externalities become irrelevant.[3]

[1]While our results still hold for package auctions with appropriate changes of definitions, we use single-item auctions for our model for simplicity of exposition, since the dimension of a type with $m$ items and $n$ bidders is $(n+1)^m$ (or $(n+1)^m - 1$ without the seller's externalities), which is already 16 for $n = 3$ and $m = 2$, for instance, and a type profile is $(n+1)^m$ by $(n+1)$. Note that, however, even in single-item auctions with externalities, multiple bidders may need to pay, as in package auctions without externalities; thus, the desirability of the core property can be applied similarly.

[2]Strictly speaking, any *one-step* deviation is impossible since the core is a one-step deviation concept. For multiple-step deviations, e.g., resale, see Section 6.

[3]For a numerical example, consider three bidders A, B, and C whose values are 9, 10, and 7 (respectively), but C would suffer a negative externality of 2 if B wins. In this case, B, who has the highest valuation, may not win any more because C is willing to pay up to 2 in order to prevent B from winning by "helping" A to win. In fact, for an efficient allocation, A should win. However, if A wins but the sum of payments of A and C is less than 10, which is B's maximum willingness to pay in order to beat A, then this outcome can be unstable or disputable. On the other hand, suppose A wins at 9 and C pays 2 (B does not pay), where payoffs of A, B, and C are 0, 0, and $-2$. Then, C rather wants to bribe A and B (by paying 0.5 each) to agree not to buy, where their payoffs are 0.5, 0.5, and $-1$. Alternatively, since B and C do not suffer an externality even if A wins, B and C can be even better off if C only bribes B by paying 0.6, where payoffs

Thus, approximate cores (Shubik and Wooders, 1983; Wooders and Zame, 1984; Kaneko and Wooders, 1989) are also empty in general. However, we still need to choose some outcome for auctions, then what can be good alternative outcomes?

All deviations are undesirable but some deviations are more undesirable than others. In fact, we can categorize deviations into two: *pay more* and *refuse to pay*. Deviations by *paying more* may be more undesirable while deviations by *refusing to pay* may be unrealistic. *Pay more*, i.e., the existence of a blocking coalition including the seller and some bidders is undesirable in terms of fairness as well as stability. Suppose the seller sold the item to a buyer at a lower price than some buyers are willing to pay together, then the latter buyers would claim that this auction is unfair, i.e., their claim can be seen as *justified envy*,[4] which is also undesirable to the seller in terms of revenue, i.e., the so-called *low revenue problem* (e.g., Ausubel and Milgrom (2006)).

This paper is the first to provide desirable alternative outcomes when the core is empty: efficient outcomes with no justified envy where any possible deviations need payment refusals that are infeasible or subject to penalty in practice. That is, we show that the core with *no payment refusals* (NPR) is nonempty and fully characterize the core (Proposition 1), i.e., if bidders cannot refuse payments, the core is nonempty.[5] In other words, it is always possible to have an outcome with *no justified envy* (NJE), i.e., no group of bidders is willing to pay more than the final price while still satisfying *individual rationality* (IR)[6] for everyone. NJE is equivalent to *group rationality* (GR) for a coalition including the seller and any subset of bidders (Lemma 2), which also implies efficiency. Thus, we can always have an efficient outcome where the seller has no incentive to deviate. As usual, the core with NPR is not single-valued in general. As a core refinement, we show that the maximum-revenue core-selecting mechanism is equivalent to the optimal mechanism of Jehiel et al. (1996) (Proposition 2). On the other hand, minimum-revenue core-selecting mechanisms

of B and C are 0.6 and $-0.6$. That is, even in this simple example, the core is empty.

[4]See Definition 2. Note that it might be better to differentiate stability from "fairness" in terms of their reference types: stability should be with respect to (true) values while fairness should be with respect to (reported) bids. For instance, suppose there are two bidders in a first-price auction where bidder 1's value is 10 and bid is 7, and bidder 2's value is 9 and bid is 8. Then, bidder 2 wins at 8. In this case, bidder 1 and the auctioneer may want to deviate, i.e., the outcome is *unstable*, but it might be unreasonable for bidder 1 to claim that this outcome is *unfair* because she had a chance to submit a higher bid, and the auction outcome followed the auction rule upon which the bidders agreed before they participated. While we always use true values in this paper, we also study incentive properties in Section 5. Note also that Goeree and Lien (2016) show that when any equilibrium auction outcome is in the core, it is equivalent to the VCG outcome.

[5]With the same example in footnote 3, for a core outcome with NPR, A wins and pays at most 9, B does not pay, C pays at most 2, but A and C pay at least 10 together. An outcome where A and C pay 10 together is a minimum-revenue core outcome, whereas an outcome where A pays 9 and C pays 2 is a maximum-revenue core outcome.

[6]IR should better be satisfied. Otherwise, the winner may pay "too much," which still satisfies NJE but is obviously undesirable.

minimize the sum of the gains of bidders by misreporting (Corollary 2). Our existence result is more positive in the following sense. Jehiel and Moldovanu (1996) use the least sharp core, the $\alpha$-core, i.e., if the $\alpha$-core is empty, all other cores are also empty. We use the same core for the main results (we also discuss other compelling core concepts in Appendix A), but we show that the $e$-core, a subset of the $\alpha$-core (i.e., which is more difficult to be nonempty), is also nonempty with NPR (Proposition 8). In addition, we do not impose any restrictions on externalities and valuations, i.e., positive externalities and negative valuations are also allowed (Appendix C). In contrast, the literature normally either assumes no positive externalities or uses normalization such as the seller neither imposes nor suffers any externalities, which is not without loss of generality (see footnote 9 for details).

Unfortunately, *refuse to pay* cannot be resolved, i.e., it is not always possible to satisfy both NJE and GR for bidders.[7] If an outcome is not GR for some bidders, then they are better off refusing to pay. For instance, they can bribe the winner not to buy (i.e., some bidders including the winner refuse to pay), or losing bidders can refuse to pay. Although this possibility of deviations might be disappointing in terms of stability, it might be fine in terms of fairness. Even though bidders may want to deviate (by refusing to pay) when an outcome is not GR (but IR), they might not claim that the auction is unfair if a payment is based on the auction rule on which they already agreed before participating. Moreover, when externalities are real concerns, it is most likely a high-stakes auction where each bidder is normally a big company, an organization, or a government. Thus, as sellers may care about their reputation, bidders may also care about their reputation, which may also affect their participation qualification in the future, and therefore bidders may not want to deviate by refusing to pay. In fact, in many auctions, once a bidder participates, its payment is a binding contract.

The deviations by refusing to pay can be mitigated or resolved in other ways. Side payments (often considered a collusion) are prohibited in many auctions. While not all deviations by payment refusals need side payments, bribing the winner or a loser who suffers the worst negative externality needs side payments. Thus, instability can also be mitigated by no side payments. On the other hand, some auctions allow a purchase cancellation with some penalty; thus, we also find the minimum fine that guarantees a nonempty core. In particular, when there is no loser's payment, the amount of the fine is reasonable in the sense that the fine is at most the maximum bid of the winner, i.e., what she agreed to pay in order to beat some bidder (Proposition 4).

Without a loser's payment, a core-selecting mechanism is impossible in general even with NPR (Proposition 5). While externalities are common in auctions where commercial

[7] However, Proposition 3 shows that GR for some bidders can further be satisfied.

bidders participate, auctions without (i.e., disallowing) a loser's payment are widely used in practice, e.g., the first-price, second-price, and English auctions. We show that, interestingly, if a loser's payment is not allowed, a core outcome no longer needs to be efficient, and NPR ensures a nonempty core (Proposition 6). Even with NPR, usual one-dimensional (bid) mechanisms do not guarantee a core outcome and have many other problems: inefficiency, low revenue, various kinds of regrets, etc. Jeong (2017) discusses these problems including a new problem called *group winner regret* (where bidders unnecessarily compete with each other and regret not losing to some other bidder) and introduces new mechanisms, the Multidimensional Second-Price (MSP) and the Multidimensional English (ME) auctions. The MSP is a core-selecting mechanism when a loser's payment and a purchase cancellation are not allowed. In the MSP, the winner cannot win at any lower price by misreporting (i.e., no regret about the price), and losers cannot be better off winning by misreporting. When there are no externalities, the MSP reduces to the standard second-price auction. Furthermore, the MSP is a unique direct mechanism that satisfies several good properties.

As in core-selecting package auctions, the Vickrey-Clarke-Groves (VCG) mechanism (Vickrey, 1961; Clarke, 1971; Groves, 1973) and the core are closely related. We show that most relevant results in the core-selecting package auction literature still hold in our model even with no payment refusals. For instance, the result that the VCG payoff is the maximum core payoff for bidders still holds even with NPR (Proposition 7), which implies that there is no core outcome without a loser's payment when the VCG has a loser's payment.

When all kinds of deviations are allowed, the core can be empty. Even when the core is empty, however, we still need to choose some alternative outcome. This paper provides desirable solutions: efficient outcomes with no justified envy where any possible deviations require payment refusals that are infeasible or subject to penalty in practice.

The remainder of this paper is organized as follows. Section 2 introduces the model and defines the core. When externalities exist, the definition of a coalitional value function (which needs to define group rationality and the core) is nontrivial since the coalitional value of losing bidders may not be zero. Section 3 shows the main results. Section 4 shows the conditions for a nonempty core when loser's payment is not allowed. Section 5 shows the relationship between the core and the VCG. Section 6 concludes the paper. Appendix A discusses various effectiveness concepts and their corresponding cores. Appendix B shows the core when no seller's deviations are allowed (Proposition 9). Appendix C allows positive externalities and negative valuations. All the main results still hold with trivial changes.

## 2 The Model

There are one indivisible item, one auctioneer (denoted by "bidder" 0), and a set of bidders $N = \{1, 2, ..., n\}$. $N^0 = N \cup 0$.[8] The type of bidder $j$ is denoted by a column vector $\mathbf{t}_j = (t_{ij})_{i \in N^0}$, where $t_{ij} \leq 0$ denotes the externality imposed on bidder $j$ when bidder $i \neq j$ wins the item, and $t_{jj} \geq 0$ denotes the bidder $j$'s own valuation of the item. In contrast to the most literature, in Appendix C, we do not impose any restriction on $t_{ij}$ for all $i, j \in N^0$. That is, (with trivial changes) all the results hold with positive externalities and negative valuations.[9] No restrictions on $t_{ij}$, however, unnecessarily hinder intuition, so we first assume nonpositive externalities and nonnegative valuations, and relegate the extension to Appendix C. The type profile of $n$ bidders and the auctioneer is denoted by a $(n+1)$-by-$(n+1)$ type matrix, $T = (t_{ij})_{i,j \in N^0}$. Let $b_{ij} \equiv t_{jj} - t_{ij}$, i.e., $j's$ maximum willingness to pay in order to beat $i$. The payoff of $j \in N$ is denoted by $u_j \equiv t_{wj} - p_j$ when her payment is $p_j$ and $w \in N^0$ wins. The seller's payoff or revenue is $p \equiv u_0 \equiv \sum_{j \in N} p_j$.

Nonparticipation of some bidders does not affect the remaining bidders' types, i.e., even if only a subset of bidders $S \subseteq N^0$ participates in an auction, $t_{ij}$ for $i, j \in S$ does not change. Each bidder's outside option is determined by the outcome of an auction, i.e., even if bidder $j$ does not participate in an auction, $u_j = t_{ij}$ if $i$ wins. The auctioneer can neither extract payments from nonparticipants, nor give them the item (the "nondumping" assumption).[10] Let $T_S$ denote the $|S|$-by-$|S|$ submatrix of $T$ that has only rows and columns $j \in S$. Without loss of generality, we assume that $t_{j0} = 0$ and $t_{0j} = 0$ for all $j \in N^0$. That is, the auctioneer neither imposes nor suffers any externalities. Thus, for simplicity, normally only the types of bidders, $T_N$, will be given in examples.

A direct auction mechanism is a pair of functions $\varphi = (x, \rho)$, $\varphi : \mathcal{T} \to (N^0, \mathbb{R}^n)$, where $x$ is the winner-determination rule with a tie-breaking rule, $\rho$ is the payment rule, and $\mathcal{T} \equiv \{(t_{ij})_{i,j \in N^0} \in \mathbb{R}^{n+1} : b_{ij} \leq \bar{b}\}$ for some sufficiently large bound $\bar{b} \in \mathbb{R}_+$.[11] An auction outcome is denoted by $(w, \mathbf{p})$, where $w \equiv x(T)$ is the winner (can be the auctioneer, i.e., no sale), and $\mathbf{p} \equiv (0, \rho(T)) \in \mathbb{R}^{n+1}$ is the payment of the auctioneer and all bidders with $p_0 \equiv 0$ (for notational simplicity in Definition 2). An auction outcome is said to be *efficient*

[8] Abusing the notation, $\{j\}$ may be written as $j$.

[9] For instance, the value of garbage may be negative for many people, but positive for recycling companies. Without loss of generality we can normalize $t_{0j} = 0$ for all $j \in N^0$ once positive externalities and negative valuations are allowed. For instance, $T = \begin{bmatrix} -1 & 2 & 0 \\ 0 & 1 & 9 \\ -8 & -8 & 7 \end{bmatrix}$ can be normalized to $T' = \begin{bmatrix} 0 & 0 & 0 \\ 1 & -1 & 9 \\ -7 & -10 & 7 \end{bmatrix}$. However, we cannot further normalize $t_{i0} = 0$ for all $i \in N^0$ without loss of generality.

[10] Jehiel and Moldovanu (1996) (i.e., the empty core result) also use the nondumping assumption, which is realistic in the auction setting.

[11] Without an upper bound $\bar{b}$, an auction (e.g., VCG) outcome can be undefined due to $\infty - \infty$.

if $w \in \arg\max_{i \in N^0} \sum_{j \in N^0} t_{ij}$. When $S^c \equiv N^0 \backslash S$, $0 \in S \subseteq N^0$, refuses to participate, the auction outcome is determined by $\varphi$ with subtype $T_S$, where $\varphi$ is implicitly defined in the subspace, $T = T_S$ and $\mathcal{T} = \mathcal{T}_S$. Abusing the notation, even when only $S$ participates, $\mathbf{p} \in \mathbb{R}^{n+1}$. When no bidders participate, the auctioneer keeps the item, and $\mathbf{p} = \mathbf{0}$.

Before defining the core, as an example of direct auction mechanisms, the Vickrey-Clarke-Groves (VCG) mechanism (Vickrey, 1961; Clarke, 1971; Groves, 1973) is provided.

**Definition 1.** For a reported type profile $T$, the VCG auction mechanism is $\varphi = (x, \rho)$, where $x(T) \in \arg\max_{i \in N^0} \sum_{j \in N^0} t_{ij}$ and $\rho_j(T) = \sum_{k \in N^0 \backslash j} t_{x(T_{-j}),k} - \sum_{k \in N^0 \backslash j} t_{x(T),k}, \forall j \in N$.

**Example 1.** Let $T_N = \begin{bmatrix} 9 & -3 & 0 \\ 0 & 7 & -2 \\ 0 & 0 & 1 \end{bmatrix}$. That is, the valuations of bidders 1, 2, and 3 are 9, 7, and 1, respectively, but bidder 2 suffers a negative externality when bidder 1 wins, i.e., $u_2 = t_{12} = -3$, and bidder 3 suffers a negative externality when bidder 2 wins, i.e., $u_3 = t_{23} = -2$. Thus, for instance, bidder 2 is willing to pay only up to $b_{32} \equiv t_{22} - t_{32} = 7 - 0 = 7$ in order to beat bidder 3, but up to $b_{12} \equiv t_{22} - t_{12} = 7 - (-3) = 10$ to beat bidder 1.

The VCG outcome is $w = 1$, $p_1 = 5 - (-3) = 8$, $p_2 = 9 - 9 = 0$, and $p_3 = 7 - 6 = 1$. That is, $\mathbf{p}_N = (8, 0, 1)$, $p = 9$, and $\mathbf{u} = (9; 1, -3, -1)$.[12] There are two interesting facts. First, there is a loser's payment. Second, in order to beat $w = 1$, bidder 2 is willing to pay up to $b_{12} = 10$, which is greater than the current price $p = 9$. That is, bidder 2 and the seller can deviate, which will be formally discussed in the rest of this section.

The core and individual rationality are defined by a *coalitional value function* (or *characteristic function*) $v(S)$ for a coalition $S$.[13] In a cooperative game $(N^0, v)$ with transferable utility, for a given payoff profile $\mathbf{u}$, if $\sum_{j \in S} u_j < v(S)$ for some $S \subseteq N^0$, then $\mathbf{u}$ is said to be *blocked* by a *blocking coalition* $S$. On the other hand, $\mathbf{u}$ is said to be *individually rational* (IR) for $j \in N^0$ if $u_j \geq v(j)$, and *group rational* (GR) for $S \subseteq N^0$ if $\sum_{j \in S} u_j \geq v(S)$.[14] The *core* is defined as the set of payoff profiles that are feasible, i.e., $\sum_{j \in N^0} u_j = v(N^0)$, and not blocked by any coalition. That is,

$$Core(N^0, v) = \left\{ \mathbf{u} \in \mathbb{R}^{n+1} : \sum_{j \in N^0} u_j = v(N^0) \text{ and } \sum_{j \in S} u_j \geq v(S) \text{ for all } S \subseteq N^0 \right\}, \quad (1)$$

but we also explicitly characterize the core in each result for intuition and easier calculations. An auction outcome is said to be in the core (or a *core outcome*) if the payoff profile of the

[12] To emphasize the seller's payoff, we use a semicolon as a separator.

[13] The definition of $v(S)$ will follow. For notational simplicity, the type is omitted, i.e., $v(S) = v(S; T)$.

[14] Since GR implies IR, when there is no ambiguity, we will often use "bidders" and "a group of bidders" to denote either one bidder or more than one bidder.

outcome is in the core. An auction mechanism is said to be *core-selecting* (or said to have the *core property*) if all outcomes are in the core.

Due to externalities, we need to define the coalitional value function carefully because the coalitional value of a coalition without the auctioneer can be nonzero. To define any coalitional value of $S$, we need to choose what strategies the outside players, $S^c = N^0 \backslash S$, would use, which is the concept of *effectiveness* in the core with externalities literature (Aumann and Peleg, 1960; Shapley and Shubik, 1969; Chander and Tulkens, 1997).

In particular, $\alpha$-effectiveness considers the worst cases that the deviators could face. That is, if all kinds of deviations are allowed,

$$v_\alpha(S) = \begin{cases} \max_{i \in S} \sum_{j \in S} t_{ij} & 0 \in S \\ \min_{i \in S^c} \sum_{j \in S} t_{ij} & 0 \notin S \end{cases}. \tag{2}$$

As in Jehiel and Moldovanu (1996) where they show the core can be empty, we use $\alpha$-effectiveness, i.e., $v(\cdot) \equiv v_\alpha(\cdot)$. That is, the core is the $\alpha$-core, and IR (or GR) is IR (or GR) by $\alpha$-effectiveness. Due to its "pessimism," the $\alpha$-core is the least-sharp core, i.e., if the $\alpha$-core is empty, all other cores are also empty. One may disagree with this pessimism, which may be a real concern, since even if the $\alpha$-core is nonempty with some restriction, a more realistic (i.e., less pessimistic) core might still be empty. Thus, in Appendix A, we discuss other compelling core concepts, and also show that the $e$-core, a subset of the $\alpha$-core (i.e., which is more difficult to be nonempty), is also nonempty with the same restriction.

For a nonempty core, we impose some restrictions, which lead to nontransferable utility and modifications of $v_\alpha(\cdot)$ in Equation (2). As in Aumann and Peleg (1960) and Aumann (1961), with nontransferable utility, the coalitional value function is not scalar in general, so the coalition value function and the core should be redefined appropriately. In our case, with main restrictions,[15] $v_\alpha(\cdot)$ still remains as "scalar" but may depend on the timing, i.e., $v_\alpha(\cdot)$ may have two different scalar values depending on whether it is for nonparticipation (before an auction) or for deviation (after an auction).

Before showing the main results, we show that the core is empty if no sale is efficient even with a positive valuation of some bidder. To avoid this trivial emptiness, we use the following "mild" assumption for the rest of this paper.

**Lemma 1.** *If* $\max_{i \in N} \sum_{j \in N} t_{ij} < 0$ *with* $t_{jj} > 0$ *for some* $j \in N$*, the core is empty.*

[15] Specifically, no purchase cancellations (Definition 3) and no payment refusals (Definition 4). In contrast, no loser's payment (Definition 5) or no seller deviations (Definition 7) makes $v_\alpha(\cdot)$ non-scalar (due to multiple winners) even for one timing. In this case, due to multiple values of $v_\alpha(\cdot)$, the core cannot be defined simply, as in Equation (1). Instead of changing Equation (1) complicatedly, we explicitly characterize the core.

*Proof.* The seller should keep the item for efficiency. Let $i = \arg\max_j t_{jj}$ (by some tie-breaking if needed), then $p$ should be at least $t_{ii}$. Otherwise, $\{0, i\}$ can block the outcome. However, if $p \geq t_{ii}$, then there exists $k$ such that $p_k > 0$. Without loss of generality, let $k = 1$. Then, bidder 1 can bribe $N\backslash 1$ not to buy, i.e., $N$ can block $\mathbf{u}$ with $(0; -(n-1) \cdot \epsilon, u_2 + \epsilon, u_3 + \epsilon, ..., u_n + \epsilon)$ for $\epsilon$ such that $0 < (n-1) \cdot \epsilon < p_k$. On the other hand, when $\max_{j \in N} t_{jj} \leq 0$, the core is $\{\mathbf{0}\}$. □

**Assumption 1.** *There exists an efficient sale, i.e.,* $\max_{i \in N} \sum_{j \in N} t_{ij} \geq 0$*, or all values are nonpositive, i.e.,* $\max_{j \in N} t_{jj} \leq 0$*.*

# 3 The existence of a nonempty core

Jehiel and Moldovanu (1996) show that the core can be empty in general. However, some deviations may be unrealistic while some deviations may be more undesirable. In fact, we can categorize deviations into two: *pay more* and *refuse to pay* as follows: any deviation by some $S \subseteq N^0$ means GR for $S$ is violated, i.e., $S$ is a blocking coalition. There are two cases: $0 \in S$ and $0 \notin S$. The former case is the *pay more* case since the seller has no incentive to deviate otherwise. The latter case is the *refuse to pay* case, which might be nontrivial to see the equivalence. If $S$ deviates, a new allocation among $S^c$ could only be equal or worse to $S$ than the current allocation; thus, for $S$ to be better off, the sum of the payments of $S$ must be strictly smaller than the current sum of the payments of $S$, i.e., some bidder in $S$ refuse to pay (at least partially). From now on, for notational simplicity, we mainly assume $0 \in S$ unless otherwise specified, i.e., $S$ denotes some subset of $N^0$ including the seller, and $S^c \equiv N^0 \backslash S$ denotes some subset of bidders.

The emptiness of the core means we cannot satisfy both no deviations by paying more (i.e., GR for $S$ with $0 \in S$) and no deviations by refusing to pay (i.e., GR for $S^c$ with $0 \in S$). Between the two deviations, paying more may be more undesirable while refusing to pay may be unrealistic. If there are bidders who are willing to pay more than the final price, they might claim that this auction is unfair, i.e., their claim can be seen as "justified envy." Thus, a blocking coalition including the seller is undesirable in terms of fairness as well as stability (and revenue). In the following definition, $t_{w'j} - (t_{wj} - p_j)$ is the maximum willingness to pay of bidder $j$ in favor of winning of $w'$ over $w$.

**Definition 2.** For an auction outcome $(w, \mathbf{p})$, $S$ $(0 \in S \subseteq N^0)$ is said to have *justified envy* if $\sum_{j \in S}(t_{w'j} - (t_{wj} - p_j)) > p$ for some $w' \in S$. Otherwise, $S$ is said to have *no justified envy* (NJE).

In the following lemma, we show that NJE is equivalent to GR for $S$. Note that NJE includes IR for the seller (i.e., $p \geq 0$) as a special case when $S = \{0\}$, which is consistent with the following lemma since GR implies IR.

**Lemma 2.** *No justified envy is equivalent to group rationality for all $S \subseteq N^0$ with $0 \in S$. In addition, no justified envy implies efficiency.*

*Proof.* For the first part, suppose it is not GR for some $S$. Then, by the definition of GR, $\sum_{j \in S} u_j = p + \sum_{j \in S}(t_{wj} - p_j) < \max_{i \in S} \sum_{j \in S} t_{ij} = v(S)$. This inequality is equivalent to $\sum_{j \in S}(t_{w'j} - (t_{wj} - p_j)) > p$, where $w' = \arg\max_{i \in S} \sum_{j \in S} t_{ij}$ (with a tie-breaking if needed). Thus, there exists justified envy. For the converse part, suppose there exists justified envy, i.e., $\sum_{j \in S}(t_{w''j} - (t_{wj} - p_j)) > p$ for some $w'' \in S$. Since $w' = \arg\max_{i \in S} \sum_{j \in S} t_{ij}$, $\sum_{j \in S}(t_{w'j} - (t_{wj} - p_j)) \geq \sum_{j \in S}(t_{w''j} - (t_{wj} - p_j)) > p$, which is equivalent to $\sum_{j \in S} u_j < v(S)$. Thus, GR for $S$ does not hold. The second part is immediate with $S = N^0$. $\square$

In the definition of justified envy, we allow $w' = 0$, which is the *bribing (the seller) not to sell* case (recall that $p_0 \equiv 0$). Of course, it might be better to differentiate these two cases, sale and no sale, because bidders may claim that the justified envy with $w' \neq 0$ is much unfairer than the *bribing not to sell* especially when the seller must sell the item. Fortunately, however, both cases (i.e., GR for $S$) do not bite for the nonemptiness of the core; thus, for notational simplicity and consistency with GR for $S$, we use justified envy to denote both cases.

Interestingly, justified envy with $w' = w$ is possible when $p_j < 0$ for some $j \notin S$. For instance, suppose $T_N = \begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix}$, $w = 1$, $p_1 = 7$, and $p_2 = -2$. Then bidder 1 alone is willing to pay more than $p = 5$, so $S = \{0, 1\}$ has justified envy with $w' = w$. We can actually show that $\mathbf{p} \geq 0$ (where $p_0 \equiv 0$ by definition) is needed for NJE.

**Lemma 3.** *For an auction outcome $(w, \mathbf{p})$ to have no justified envy, $\mathbf{p} \geq 0$.*

*Proof.* Suppose $p_j < 0$ for some $j \in N$, then $S = N^0 \backslash j$ can block this: the seller can pay $\epsilon/(n-1)$ additionally to each bidder in $S$ for $0 < \epsilon < p_j$ and make $p_j = 0$. $\square$

While no blocking coalition of bidders (i.e., GR for $S^c$) cannot be satisfied together with GR for $S$ in general, deviations by bidders require payment refusals, which may be unrealistic. In many auctions, the seller checks credit or budget as a bid validation to prevent a default, and a winning is a binding contract. Even when a purchase cancellation is allowed, there usually exists a cancellation fee.

**Definition 3.** *No purchase cancellations* (NPC) mean the winner cannot cancel a purchase.[16] The *cancellation fee* is a fee that the winner needs to pay with a purchase cancellation.

Unfortunately, however, NPC does not guarantee a nonempty core since losers may still deviate on their own. While GR is satisfied for losers without a payment (Lemma 4), GR may not be satisfied if a loser with a payment is included (Example 2).

**Lemma 4.** *For an auction outcome* $(w, \mathbf{p})$*, group rationality is satisfied for losing bidders with nonpositive payments, i.e.,* $\sum_{j \in S} u_j \geq v(S)$ *for all* $S \subseteq N \backslash w$ *with* $p_j \leq 0$ *for all* $j \in S$*.*[17]

*Proof.* $\sum_{j \in S} u_j = \sum_{j \in S}(t_{wj} - p_j) \geq \sum_{j \in S} t_{wj} \geq \min_{i \in S^c} \sum_{j \in S} t_{ij} = v(S)$. □

**Example 2.** For $T_N$ in Example 1, assume NPC. By NJE, which also implies efficiency (Lemma 2), $w = 1$ and $p \geq 10 = b_{12}$. By IR for bidder 1, $p_1 \leq 9$; thus, $p_3 \geq 1$. Even though bribing bidder 1 not to buy is impossible due to NPC, the following deviation is possible: for $S^c = \{2, 3\}$, $\sum_{j \in S^c} u_j \leq -3 + (-1) < -3 = \min_{i \in S} \sum_{j \in S^c} t_{ij} = v(S^c)$; thus, GR for $S^c$ is not satisfied.

Although the above result (i.e., the existence of a blocking coalition of bidders) might be disappointing in terms of stability, it may be acceptable compared to justified envy (i.e., the existence of a blocking coalition including the seller) in terms of fairness. In Example 2, a loser, bidder 3, also needs to pay, but a loser's payment is not so unusual in auctions with externalities (e.g., Jehiel et al. (1996)). Losers are willing to pay in order to avoid a bad allocation. Thus, even though bidders may want to deviate (by refusing to pay) when an outcome is not GR (but IR), they may not claim that the auction is unfair if a payment is based on an auction rule on which they already agreed before participating. Moreover, if bidders explicitly concern about externalities, it is most likely a high-stakes auction. Thus, bidders may also care about their reputation, which may also affect their participation qualification in the future, and therefore bidders may not want to deviate by refusing to pay. Since NPC is not enough for a nonempty core, we introduce the following stronger restriction than NPC.

[16] To avoid a pathological core outcome (i.e., the winner pays "too much") due to NPC, we assume that the outcome is IR for the winner, assuming a "purchase cancellation" is allowed. That is, NPC requires $v_\alpha(S)$ in Equation (2) to have the following modification: $v_\alpha(S) = \min_{i \in S^c} \sum_{j \in S} t_{ij} - p_w$ if $0 \notin S, w \in S \neq \{w\}$ (i.e., $w$ still needs to pay even with deviations). As implied by referring to $w$, these modifications of $v_\alpha(\cdot)$ are only for deviations (after an auction). For nonparticipation (before an auction), $v_\alpha(\cdot)$ remains the same as in Equation (2). Note also that "$S \neq \{w\}$" enables IR for $w$, which is already implied by Equation (2).

[17] This result holds even when the seller cannot deviate (see Appendix B for the core with no seller deviations) as follows: the same winner still wins when $S$ does not participate; thus, $\sum_{j \in S} u_j = \sum_{j \in S}(t_{wj} - p_j) \geq \sum_{j \in S} t_{wj} = v(S)$.

**Definition 4.** *No payment refusals* (NPR) mean bidders cannot refuse a payment.[18]

NPR implies NPC. We show that NPR guarantees a nonempty core. That is, a group of bidders (not an individual) may want to deviate (which requires a payment refusal), but they never have justified envy. In other words, we can always have an outcome that might be unstable but fair in the sense that it is free of justified envy. The following proposition shows this result (see also Example 3 with Figure 1). Note that by NJE (3), $w \in \arg\max_{i \in N^0} \sum_{j \in N^0} t_{ij}$, i.e., efficient allocation (Lemma 2), and $\mathbf{p} \geq 0$ (Lemma 3).

**Proposition 1.** *In auctions with no payment refusals, the core is nonempty and is* $\{(p; t_{w1} - p_1, t_{w2} - p_2, ..., t_{ww} - p_w, ..., t_{wn} - p_n)\}$, *where* $w$ *and* $\mathbf{p}$ *satisfy*

$$\sum_{j \in S} (t_{w'j} - (t_{wj} - p_j)) \leq p, \forall w' \in S, \forall S \subseteq N^0, 0 \in S \quad \text{(NJE or GR for } S\text{)} \tag{3}$$

*and*

$$t_{wj} - p_j \geq \min_{i \in N^0 \setminus j} t_{ij}, \forall j \in N. \quad \text{(IR for } N\text{)} \tag{4}$$

*Proof.* It is sufficient to show that both GR for $S$ (with $0 \in S$) and GR for $S^c$ hold. First, GR for $S^c$ holds once IR holds: GR for bidders including the winner (i.e., impossibility of bribing the winner not to buy) holds by NPC (less restrictive than NPR). GR for losers without a payment holds by Lemma 4 even without NPR. Finally, by NPR (as opposed to Proposition 9 or Example 2), GR for losers even with a payment also holds, since $S^c$ still needs to pay with a deviation (which leads to a reallocation among $S$) where they can only be weakly worse off.

Now we will prove GR for $S$, i.e., NJE, also holds. But (3) is the NJE condition; thus, we only need to show the existence of $(w, \mathbf{p})$ that satisfies NJE (3) and IR (4). The proof of the existence can start from the VCG (or can be proven directly by Proposition 2): from the VCG outcome $(w, \mathbf{p}^{VCG})$, which is efficient (implied by NJE) and IR, we will adjust $\mathbf{p}^{VCG}$ to find $\mathbf{p}$ such that there is NJE. That is, we only need to show that there is no blocking coalition $S \subset N^0$ $(0 \in S)$ (which leads to a new allocation to $w' \in S$) such that the payments of $S^c$ (with the original allocation) cannot be adjusted to make the payoffs have NJE (which implies IR of $S$) while maintaining IR of $S^c$.

Suppose such a blocking coalition $S$ exists. By justified envy, $\sum_{j \in S}(t_{w'j} - (t_{wj} - p_j)) > p$.

[18] As in NPC, to avoid a pathological core outcome (i.e., pay "too much") due to NPR, we assume that the outcome is IR, assuming a "payment refusal" is allowed. That is, NPR requires $v_\alpha(S)$ in Equation (2) to have the following modification: $v_\alpha(S) = \min_{i \in S^c} \sum_{j \in S} t_{ij} - \sum_{j \in S} p_j$ if $0 \notin S, |S| > 1$ (i.e., $S$ still needs to pay even with deviations). Note that this definition can be used for both nonparticipation (before an auction) and deviation (after an auction), if we assume $p_j = 0, \forall j \in N$ before an auction.

In addition, the impossibility of the payment adjustment which maintains IR of $S^c$ means

$$\sum_{j\in S}(t_{w'j} - (t_{wj} - p_j)) - p > \sum_{j\in S^c}((t_{wj} - p_j) - \min_{i\in S} t_{ij}),^{19} \tag{5}$$

which gives

$$\sum_{j\in N} t_{w'j} > \sum_{j\in N} t_{wj} + \left(\sum_{j\in S^c} t_{w'j} - \sum_{j\in S^c} \min_{i\in S} t_{ij}\right).$$

Since $\sum_{j\in S^c} t_{w'j} - \sum_{j\in S^c} \min_{i\in S} t_{ij} \geq 0$, the last inequality implies $\sum_{j\in N} t_{w'j} > \sum_{j\in N} t_{wj}$, which contradicts the efficiency of the original allocation. □

Since NPR only prohibits certain deviations but not introduce new deviations, the core with NPR includes the core without NPR (even though the latter might be empty). In most cases, the core (with NPR) is not single-valued. Before refining the core, due to NPR, we check if the core is still compact and convex.

**Lemma 5.** *The core in Proposition 1 is a compact convex set.*

*Proof.* Each constraint of (3) or (4) corresponds to a closed convex set. The intersection of any collection of closed sets is closed, and the intersection of any collection of convex sets is convex. The core is bounded by (4); thus, together with closedness, the core is compact. □

Since the core is compact, by the extreme value theorem, there exist minimum and maximum revenues of the core. We first show that the maximum-revenue core-selecting mechanism is equivalent to the optimal (revenue) mechanism in Jehiel et al. (1996) as follows.

**Proposition 2.** *The maximum-revenue core outcome in Proposition 1 is unique (up to ties). The core is* $\{(p; t_{w1}-p_1, t_{w2}-p_2, ..., t_{ww}-p_w, ..., t_{wn}-p_n)\}$, *where* $w \in \arg\max_{i\in N^0} \sum_{j\in N^0} t_{ij}$ *and* $p_j = t_{wj} - \min_{i\in N^0\setminus j} t_{ij}$, $\forall j \in N$.

*Proof.* For any core outcome, each $p_j$ is bounded above by (4), the IR constraint. Choose each $p_j = t_{wj} - \min_{i\in N^0\setminus j} t_{ij}$, i.e., the maximum that still satisfies IR. Then, $p$ is the maximum revenue of the core once NJE is satisfied because no one can increase the payment. Now suppose there exists justified envy. However, no one can increase the payment, a contradiction. □

[19] The LHS of (5), $\sum_{j\in S}(t_{w'j} - (t_{wj} - p_j)) - p$, is the amount that $S$ is willing to pay in addition to the current payment $p$. In other words, the LHS is the minimum amount that $S^c$ needs to pay additionally in order to prevent the blocking. On the other hand, the RHS of (5) is the maximum amount that $S^c$ can pay additionally while maintaining IR.

In contrast, the minimum-revenue core outcome is usually not unique. The minimum-revenue core has good properties, e.g., minimizing the sum of the gains of bidders by misreporting (see Corollary 2). Finding the minimum-revenue core can be done in the same way as in minimum-revenue core-selecting package auctions (Day and Raghavan, 2007; Erdil and Klemperer, 2010; Day and Cramton, 2012). Note also that in the minimum-revenue core, we can further satisfy GR (assuming payment refusals are allowed) for bidders who increased their payment to prevent the blocking, which improves stability (if payment refusals are allowed) or participation incentive (if payment refusals are not allowed). In fact, this result holds stronger (i.e., GR may hold even when they pay more than the payment in the minimum-revenue core) as follows.

**Proposition 3.** *In auctions with no payment refusals, if the VCG outcome has justified envy by $B \subset N^0$ which leads to the minimum-revenue core, then the core that also satisfies group rationality for $B^c$ (assuming payment refusals are allowed) is nonempty and is $\{(p; t_{w1} - p_1, t_{w2} - p_2, ..., t_{ww} - p_w, ..., t_{wn} - p_n)\}$, where $w$ and $\mathbf{p}$ satisfy*

$$\sum_{j \in S} (t_{w'j} - (t_{wj} - p_j)) \leq p, \forall w' \in S, \forall S \subseteq N^0, 0 \in S, \quad \text{(NJE or GR for } S\text{)}$$

$$t_{wj} - p_j \geq \min_{i \in N^0 \setminus j} t_{ij}, \forall j \in N, \quad \text{(IR for } N\text{)}$$

*and*

$$\sum_{j \in B} (t_{w'j} - (t_{wj} - p_j)) - p \leq \sum_{j \in B^c} (t_{wj} - p_j) - \min_{i \in B} \sum_{j \in B^c} t_{ij}. \quad \text{(GR for } B^c\text{)}$$

*Proof.* The first two conditions are the same as in Proposition 1. Thus, we only need to show the last condition holds. As in Proposition 1, suppose GR for $B^c$ cannot hold, i.e.,

$$\sum_{j \in B} (t_{w'j} - (t_{wj} - p_j)) - p > \sum_{j \in B^c} (t_{wj} - p_j) - \min_{i \in B} \sum_{j \in B^c} t_{ij},$$

which gives

$$\sum_{j \in N} t_{w'j} > \sum_{j \in N} t_{wj} + \left( \sum_{j \in B^c} t_{w'j} - \min_{i \in B} \sum_{j \in B^c} t_{ij} \right).$$

Since $\sum_{j \in B^c} t_{w'j} - \min_{i \in B} \sum_{j \in B^c} t_{ij} \geq 0$, the last inequality implies $\sum_{j \in N} t_{w'j} > \sum_{j \in N} t_{wj}$, which contradicts the efficiency of the original allocation. □

The following example and Figure 1 show the cores in Propositions 1, 2, and 3.

**Example 3.** For $T_N$ in Example 1, the core with NPR (Proposition 1) is $\{(p; 9-p_1, -3, -p_3) : p_1 + p_3 \geq 10, p_1 \leq 9, p_3 \leq 2\}$, a shaded region in Figure 1. The maximum-revenue core

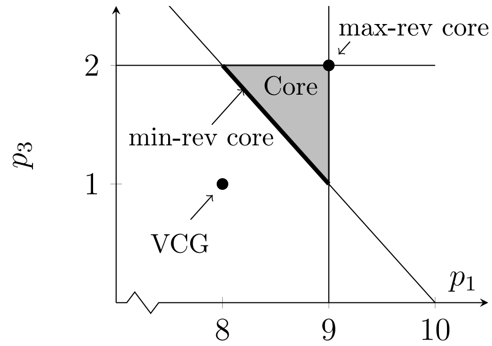


Figure 1: The core in Example 3

(Proposition 2) is $\{(11; 0, -3, -2)\}$, i.e., $w = 1$ and $p_N = (9, 0, 2)$. The minimum-revenue core is $\{(p; 9 - p_1, -3, -p_3) : p_1 + p_3 = 10, p_1 \leq 9, p_3 \leq 2\}$, a thick line in Figure 1. In this example, GR for $B^c = \{1, 3\}$ holds even in the maximum-revenue core (Proposition 3). However, this is not true in general since the worst allocations for each individual $j \in B^c$ can be different from the worst allocation for the group $B^c$.

In many auctions, side payments (i.e., monetary transfers among bidders) are often considered as a kind of collusion and therefore prohibited. Bribing the winner or a loser who suffers the worst negative externality requires side payments. One might think that no side payments imply no deviations by payment refusals, but this is not true as follows.

**Example 4.** Let $T_N = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 10 & -4 \\ 0 & -4 & 10 \end{bmatrix}$. If payment refusals are not allowed, $w = 1$ and $p_N = (7, 3, 3)$ is a core outcome. However, if payment refusals are allowed, bidders 2 and 3 can be better off refusing to pay without side payments.

While GR for $S^c$ cannot be satisfied without NPR, between the two cases, $w \in S^c$ and $w \notin S^c$, the violation of GR for $S^c$ for the former case (i.e., bribing the winner not to buy) might be more undesirable especially when the seller must sell the item. In addition, once the winner cancels a purchase, it might be unreasonable to enforce a loser's payment which was to avoid a certain undesirable allocation, i.e., if a purchase is canceled, losers may worry about undesirable future reallocations. Nevertheless, some auctions allow a purchase cancellation with some penalty, so we find the necessary purchase cancellation fee that guarantees a nonempty core with NPR by losers.

**Lemma 6.** *For an auction outcome $(w, \mathbf{p})$ that satisfies no justified envy, individual rationality, and no payment refusals by losers, the minimum cancellation fee that makes the outcome in the core is $f^* = p_w - t_{ww} + \sum_{j \in N \setminus w}(p_j - t_{wj})$.*

*Proof.* By Lemma 3, NJE implies $\mathbf{p} \geq 0$. Thus, the maximum amount of the bribe that $w$ can receive by not buying is $\sum_{j\in N\setminus w}(p_j - t_{wj}) - \epsilon$. By Proposition 1, if a purchase cancellation is never profitable, the outcome is in the core. For a purchase cancellation to be unprofitable, the fine $f$ should satisfy $u_w = t_{ww} - p_w > \sum_{j\in N\setminus w}(p_j - t_{wj}) - \epsilon - f$, which leads to $f^* = p_w - t_{ww} + \sum_{j\in N\setminus w}(p_j - t_{wj})$. □

Of course, too large a cancellation fee trivially has the same effect as NPC. Thus, the fine should also be a reasonable amount to bidders. One might think a fine up to the valuation is sufficient, which is not true as follows.

**Example 5.** Let $T_N = \begin{bmatrix} 7 & -3 & 0 \\ -2 & 9 & -5 \\ 0 & 0 & 1 \end{bmatrix}$. By NJE, $w = 1$ and $p \geq 12 = b_{12}$. By IR for bidder 2, $p_2 = 0$. By IR for bidder 3, $p_3 \leq 5$ and therefore $p_1 \geq 7$ and $u_1 = 7 - p_1$. However, if bidders 2 and 3 bribe bidder 1 not to buy, the new payoff is $u_1' = (-t_{12} + p_3 - \epsilon) - t_{11} = p_3 - 4 - \epsilon$ if the cancellation fee is $t_{11}$, i.e., bidder 1's valuation. Because $p_1 + p_3 \geq 12$, $u_1' > u_1$.

In the above example, if the penalty is $\max_{i\in N} b_{i1}$, the cancellation will be unprofitable. A penalty up to the maximum bid against others might be reasonable since it is the amount that the winner agreed to pay in order to beat some bidder. Unfortunately, however, this amount is also insufficient in general, i.e., IR for the winner cannot be satisfied. The intuition is if some losers pay in order to avoid the worst allocation even though they also suffer negative externalities due to the current winner, then their bribe can be large. However, this intuition also provides one useful sufficient condition as follows.

**Proposition 4.** *For an auction outcome $(w, \mathbf{p})$ that satisfies no justified envy, individual rationality, and no loser's payment, the minimum cancellation fee that makes the outcome in the core is $f^* \leq \max_{i\in N^0\setminus w} b_{iw}$, and this inequality is tight.*

*Proof.* By IR for $w$, $p_w \leq \max_{i\in N^0\setminus w} b_{iw}$. By Lemma 6, $f^* = p_w - t_{ww} + \sum_{j\in N\setminus w}(p_j - t_{wj}) = p_w - \sum_{j\in N} t_{wj} \leq \max_{i\in N^0\setminus w} b_{iw} - \sum_{j\in N} t_{wj} \leq \max_{i\in N\setminus w} b_{iw}$. The last inequality holds since $\sum_{j\in N} t_{wj} \geq 0$ by the efficient sale assumption (Assumption 1). The tightness can be shown by an example. For $T_N = \begin{bmatrix} 7 & -2 & -5 \\ -5 & 7 & -2 \\ -2 & -5 & 7 \end{bmatrix}$, let the winner $w = 1$ (by a tie-breaking). Then, by IR for bidder 1 and NJE, $12 = b_{13} \leq p_1 \leq \max_{i\in N^0\setminus 1} b_{i1} = b_{21} = 12$ and $u_1 = -5$. Bidders 2 and 3 are willing to bribe bidder 1 not to buy by paying up to $2 + 5 = 7$. Thus, for not buying to be unprofitable, $7 - f^* \leq -5$ and therefore, $f^* \geq 12 = b_{21} = \max_{i\in N^0\setminus 1} b_{i1}$. □

# 4 The core with no loser's payment

While externalities are common in auctions where commercial bidders participate, auctions without a loser's payment are widely used. In such auctions, the fine in Proposition 4 might be reasonable since it is at most the amount that the winner agreed to pay in order to beat some bidder. Unfortunately, however, Proposition 4 does not imply that such an outcome exists when a loser's payment is allowed. That is, when a loser's payment is allowed, there is a case where NJE and IR cannot be simultaneously satisfied without a loser's payment. In other words, without a loser's payment, the core can be empty even with NPR.[20]

**Proposition 5.** *Without a loser's payment, there exists no core-selecting mechanism even with no payment refusals.*

*Proof.* For $T_N$ in Example 1, $w = 1$ by efficiency. Without a loser's payment, $p_1 \geq b_{12} = 10$ by NJE. Thus, $u_1 \leq -1$, which violates IR for bidder 1.

Proposition 7 (i.e., even with NPR, there is no core outcome where the payment of any bidder is less than the VCG payment) is an alternative proof. The intuition is if the VCG outcome has a loser's payment but is not in the core, then to make this outcome in the core, we need to increase the payment (of $S^c$ in Proposition 1) for NJE. □

When a loser's payment is allowed, core-selecting mechanisms require a loser's payment. Many auction mechanisms in practice, however, do not allow a loser's payment from the beginning, e.g., the first-price auction. We first clarify that an auction with no loser's payment does not allow a loser's payment at all for all possible type profiles, but still allows side payments (in particular, a loser's "payment" to other bidder) as follows.

**Definition 5.** An auction mechanism $(x, \rho)$ with *no loser's payment* (NLP) means $\rho_j = 0, \forall j \in N$ with $j \neq i$.[21]

Interestingly, if we do not allow a loser's payment (as well as purchase cancellations, which might be reasonable since a deviation by a purchase cancellation requires side payments, i.e., a loser's "payment" to bidders), then a core outcome no longer needs to be efficient. An inefficient outcome cannot be blocked if it requires a loser's payment. In other words, the only possible justified envy is a *pairwise blocking*, where a pair consists of one of the bidders and the seller. For instance, for $T_N$ in Proposition 5, let $w = 2$ with $p_2 = 9$. Although this

[20]Due to no loser's payment, NPR is equivalent to NPC, but we use NPR in parallel with Proposition 1 where NPR guarantees a nonempty core.

[21]NLP requires $v_\alpha(S)$ in Equation (2) to have the following modification: $v_\alpha(S) = \left\{\sum_{j \in S} t_{ij} : \max_{k \neq i} b_{ki} \geq \max_{j \neq i} b_{ij}, i \in S\right\}$ if $0 \in S$. Note that multiple winners (denoted by $i$) can exist even without ties in $\sum_{j \in S} t_{ij}$, and a core outcome (i.e., the allocation for $S = N^0$) need not be efficient.

is an inefficient allocation so bidders 1 and 3 are willing to pay up to $9+2$, bidder 1 alone can only pay up to 9 due to IR. Thus, this inefficient outcome cannot be blocked by $\{0,1,3\}$ or any $S \subseteq N^0 \setminus \{2\}$. That is, with NLP, NJE no longer implies efficiency (as opposed to Lemma 2, which was without NLP) while it still implies $p > 0$ (Lemma 3), i.e., IR for the seller. As expected from the fact that NLP and NPR remove the GR constraint for $S^c$, they also enable a nonempty core as follows.

**Proposition 6.** *In auctions with no loser's payment, the core can be empty. In auctions with no loser's payment and no payment refusals, the core is nonempty. The core is* $\{(p; t_{w1}, t_{w2}, ..., t_{ww} - p, ..., t_{wn})\}$, *where* $w \in N$ *and* $p$ *satisfy*

$$b_{wj} \leq p, \forall j \in N^0 \quad \text{(NJE or no blocking pair)} \tag{6}$$

*and*

$$p \leq \max_{i \in N^0 \setminus w} b_{iw}. \quad \text{(IR for the winner)} \tag{7}$$

*Proof.* The first part can be shown by an example. For $T_N$ in Proposition 5, when a loser's payment is not allowed, bidder 2 should win and $9 \leq p \leq 10$. Thus, $-3 \leq u_2 \leq -2$. But, bidder 3 is willing to pay up to $-t_{23} = 2$ in order to bribe bidder 2 not to buy. Thus, every $j \in N$ can be better off by $\mathbf{u}'_N = (\delta, -2+\delta, 2-2\delta)$ for $0 < \delta < 2$. In fact, GR for $S^c = \{2,3\}$ is also not satisfied since $\sum_{j \in S^c} u_j \leq -2 + (-2) < -3 = \min_{i \in S} \sum_{j \in S^c} t_{ij}$.

For the second part, due to NLP, the condition for GR for $S$ with $0 \in S$ (i.e., NJE), which was (3) with a loser's payment, becomes (6), i.e., *no blocking pair* or *pairwise stability*.[22] Note that (6) also includes $p \geq 0$, i.e., IR for the seller, which is consistent with (3), i.e., GR for $S$ includes IR for $S$. On the other hand, GR for $S^c$ with $w \notin S^c$ (i.e., losers) holds by Lemma 4. GR for $S^c$ with $w \in S^c$ holds by NPC once IR for the winner is satisfied. The nonemptiness is obvious by construction, e.g., removing bids sequentially in a *bid graph*, as in Jeong (2017). □

**Corollary 1.** *In auctions with no loser's payment, the minimum cancellation fee that guarantees a nonempty core is* $f^* \leq \max_{i \in N^0 \setminus w} b_{iw}$, *and this inequality is tight.*

As shown in Proposition 6, in the core with NLP and NPR, not only $p$ but also $w$ might not be unique, which raises a question: which $w$ and $p$ would be a good choice for an auction mechanism. Jeong (2017) introduces a new core-selecting (with NLP and NPR) mechanism, MSP (Multidimensional Second-Price) auction, which is free of various kinds of regrets (e.g.,

[22]Although we only consider a *seller-bidder* pair, pairwise stability also includes no *bidder-bidder* blocking pair. However, a deviation by a bidder-bidder pair is impossible when a loser's payment is not allowed.

the winner cannot win at any lower price by any misreport, losers cannot be better off winning by any misreport) including *group winner regret*.

# 5 The core and the VCG

When the core is nonempty, it is usually not single-valued, and even a minimum-revenue core outcome is usually not unique. While the maximum-revenue core outcome is unique (Proposition 2), minimum-revenue core-selecting mechanisms have good incentive properties, as will be shown in Corollary 2. If we want to make a core-selecting mechanism that has a unique outcome (up to ties), the VCG can be a starting point. The VCG itself is neither core-selecting, pairwise stable, nor free of a loser's payment, which can easily be shown by Example 1. Moreover, the VCG is not core-selecting even with NPR, i.e., justified envy may exist. As in core-selecting package auctions, however, we can choose the VCG (i.e., efficient) allocation and then change payments so that the outcome could be in the core.

As shown in Proposition 5, there is no core-selecting mechanism free of a loser's payment. Thus, it would be interesting to see when a loser's payment is necessary for a core outcome. We will show that when an outcome of the VCG has a loser's payment, there is no core outcome free of a loser's payment even with NPR. In fact, this holds even stronger: even with NPR, the payment of any bidder (including the winner) in any core outcome is at least the VCG payment, i.e., the VCG payoff is the maximum core payoff for bidders.

**Proposition 7.** *Regardless of no payment refusals, there is no core outcome where the payment of any bidder is less than the VCG payment.*

*Proof.* We first prove the case where payment refusals are allowed. When there are no externalities, it is well known that the VCG payoff can be written as $u_j^{VCG} = v(N^0) - v(N^0\backslash j)$ for $j \in N$, which implies $u_0^{VCG} = v(N^0) - \sum_{j\in N} u_j^{VCG}$. In fact, this holds even when externalities exist as follows: the VCG payment rule (Definition 1) is $\rho_j(T) = \sum_{k\in N^0\backslash j} t_{x(T_{-j}),k} - \sum_{k\in N^0\backslash j} t_{x(T),k} = v(N^0\backslash j) - \sum_{k=N^0\backslash j} t_{wk}$ for $j \in N$. Thus, $u_j^{VCG} = t_{wj} - v(N^0\backslash j) + \sum_{k\in N^0\backslash j} t_{wk} = v(N^0) - v(N^0\backslash j)$ for $j \in N$. Thus, the VCG payoff is the maximum core payoff for bidders (e.g., by Theorem 5 of Ausubel and Milgrom (2002)) as follows: suppose $u_j > v(N^0) - v(N^0\backslash j)$ for some $j \in N$. Then, $\sum_{k\in N^0\backslash j} u_k = v(N^0) - u_j < v(N^0\backslash j)$; thus, $N^0\backslash j$ blocks this outcome, i.e., GR for $S = N^0\backslash j$ does not hold.

For NPR case, by Lemma 2 (i.e., the equivalence between NJE and GR for all $S \subseteq N^0$ with $0 \in S$), the fact that GR for $S = N^0\backslash j$ does not hold implies justified envy exists. Then, by Proposition 1 which requires NJE, $\mathbf{u}$ is not a core payoff profile. $\square$

**Corollary 2.** *Regardless of no payment refusals, minimum-revenue core-selecting mechanisms minimize the sum of the gains of bidders by misreporting.*

In the same spirit as the Rural Hospital Theorem (i.e., in any stable matching, the set of hospitals that are unmatched is the same) in the matching literature (e.g., Roth (1986); Hatfield and Milgrom (2005); Kojima (2012)), one may have a hope that the following holds: in any core outcome, the set of losers with a payment is the same. Unfortunately, however, this is not true. First, even with NPR (just for the existence of a nonempty core), a VCG outcome free of a loser's payment does not necessarily mean the existence of a core outcome without a loser's payment. Furthermore, the set of bidders with a loser's payment can be different in each core outcome (which also implies the previous statement), which can be easily shown by an example. The intuition is if the VCG outcome is not in the core, then to make this outcome in the core, we need to increase the payment of $S^c$ in Proposition 1 for NJE, but $S^c$ may include losers $i$ and $j$ without a payment where there is no choice but to increase at least one of $p_i$ and $p_j$. If increasing only one of $p_i$ and $p_j$ is sufficient, we have a freedom to choose; thus, the set of losers that have a payment can be different. All we can say is that if bidders want a core outcome, then any bidder cannot blame a loser's payment when the VCG also has a payment.

Since $u_j^{VCG} = v(N^0) - v(N^0 \backslash j)$ for $j \in N$, other relevant results in core-selecting package auctions (e.g., Ausubel and Milgrom (2002); Day and Raghavan (2007); Day and Milgrom (2008); Erdil and Klemperer (2010); Day and Cramton (2012)) also hold for auctions with externalities: When the VCG outcome is in the core, it is the unique bidder optimal allocation. The VCG payoff is in the core if and if only the coalitional value function is buyer-submodular.

While the VCG and the core are closely related, the VCG itself may not be desirable to use in auctions with externalities. The VCG fails to have the core property even with NPR. In addition, Jeong (2016) shows that there exist shill bidding (i.e., submitting additional "fake" bids) strategies that weakly dominate the truthful bidding. That is, as opposed to the shill bidding in package auctions, bidders have no risk (in terms of payoff, not legal issues) to use the shill bidding.

# 6 Conclusion

While the core can be empty in auctions with externalities, we showed that the core is nonempty if payment refusals are not allowed. Some bidders (not an individual) might be better off refusing to pay, but there is no case where the winning price is lower than the amount that some bidders are willing to pay. That is, we can always have an efficient outcome that might be unstable (by refusing to pay) but "fair" in the sense that it is free of justified envy.

Furthermore, since a deviation requires a payment refusal, if the seller can enforce the payment (e.g., by a deposit), there is no stability problem *per se*. Although, instability (assuming payment refusals are allowed) may affect the bidder's incentive to participate, we showed that individual rationality for everyone and group rationality for some can still be satisfied. In reality, moreover, if participants concern about externalities, it is most likely a high-stakes auction, and externalities are estimates at best, which can change according to the future environment. For instance, if the winner has the *winner's curse*, then this bidder may not impose large negative externalities as expected. Thus, not participating at all relying on the estimated externalities might be undesirable.

Another unrealistic deviation might be the seller's deviation. In many auctions, the seller "cannot" deviate. However, even if the seller's deviation is not allowed, the existence of the incentive to deviate may still be undesirable since it implies the existence of justified envy. Moreover, we can always have an outcome free of justified envy. Nevertheless, the core with no seller's deviations might be theoretically interesting; thus, it is shown in Appendix B.

The core is a one-step deviation concept, and we did not consider resales after a deviation. In practice, resales are often prohibited to prevent a delay, which may lead to social welfare loss. For instance, in spectrum license auctions where the seller is the government, if some bidder wins and tries to resale for a gain, this causes a delay, which hinders benefits from the new technology. Thus, resales are often prohibited.[23] In addition, if we allow resales (but without commitment) efficiency is impossible even with a loser's payment, as shown in Jehiel and Moldovanu (1999). Moreover, it might be unreasonable for bidders to claim that an outcome is unfair compared to an outcome that needs a multi-step deviation. Thus, the core (i.e., no one-step deviation) may be reasonable enough in terms of fairness. Nevertheless, considering resales would still be interesting, e.g., it might be possible that some stable outcomes can be attained with a smaller cancellation fee. With resale, alternative stability concept should be used, e.g., farsighted coalitional stability (Harsanyi, 1974; Chwe, 1994; Ray and Vohra, 2015).

# A Other effectiveness concepts

While we use the $\alpha$-effectiveness concept for our results, as in Jehiel and Moldovanu (1996), one may disagree with $\alpha$-effectiveness, since it might be too "pessimistic," especially in auc-

[23] In some spectrum license auctions, the government requires the winner to invest in the infrastructure over time to be able to utilize the spectrum and imposes a fine if the requirement has not met.

tions, since it implies that $S^c$ makes the worst outcome for $S$. For the negative result (i.e., the emptiness of the core in Jehiel and Moldovanu (1996)), the $\alpha$-core is the best (i.e., the most negative result). For the positive result, however, this disagreement may be a real concern, since even if the $\alpha$-core is nonempty with NPR, a more realistic (i.e., less pessimistic) core might still be empty. In this section, we not only discuss other compelling core concepts, but also show that one reasonable core, a subset of the $\alpha$-core (i.e., which means more difficult to be nonempty), is also nonempty with NPR.

Due to externalities, to define any coalitional value function of $S$, we need to choose what strategies the outside players, $S^c = N^0 \backslash S$, would use, which is the concept of *effectiveness* in the core with externalities literature (Aumann and Peleg, 1960; Shapley and Shubik, 1969; Chander and Tulkens, 1997). For a rigorous treatment, we first define a two-stage extended game $\Gamma$, where in the first stage, all the bidders decide to participate or not simultaneously, and in the second stage, the auction runs with the bidders who decided to participate.

In particular, $\alpha$ and $\beta$-effectiveness consider the worst cases that the deviators could face. In general, the coalitional value functions by *$\alpha$-effectiveness* and *$\beta$-effectiveness* (denoted by $v_\alpha$ and $v_\beta$, respectively) are defined as follows. Let $\Sigma_j$ be the strategy set of player $j \in N^0$, and $\Sigma_S = \prod_{j \in S} \Sigma_j$ for $S \subseteq N^0$. Then, $v_\alpha(S) = \max_{\sigma_S \in \Sigma_S} \min_{\sigma_{-S} \in \Sigma_{-S}} \sum_{j \in S} u_j(\sigma_S, \sigma_{-S})$, and $v_\beta(S) = \min_{\sigma_{-S} \in \Sigma_{-S}} \max_{\sigma_S \in \Sigma_S} \sum_{j \in S} u_j(\sigma_S, \sigma_{-S})$, where $u_j(\sigma_S, \sigma_{-S})$ is the payoff of $j$ when the coalition $S$ plays $\sigma_S$ and the outsiders $S^c$ play $\sigma_{-S}$. In our model, the $\alpha$-core and the $\beta$-core are the same.

**Lemma 7.** *The $\alpha$-core and the $\beta$-core coincide, i.e. $v_\alpha(S) = v_\beta(S)$ for all $S \subseteq N^0$.*

*Proof.* Because the auctioneer has the item, for $S$ such that $0 \in S$, $\Sigma_{-S} = \emptyset$. Now, if $0 \in S$, then $\Sigma_{-S} = \emptyset$. Thus, $v_\alpha(S) = v_\beta(S) = \max_{i \in S} \sum_{j \in S} t_{ij}$. Likewise, if $0 \notin S$, then $\Sigma_S = \emptyset$. Thus, $v_\alpha(S) = v_\beta(S) = \min_{i \in S^c} \sum_{j \in S} t_{ij}$. □

Again, $\alpha$-effectiveness might be too pessimistic. It might more reasonable that players in $S^c$ play for their own benefit, which is the main idea of *$\gamma$-effectiveness*. In particular, $\gamma$-effectiveness assumes that players in $S^c$ do not take particular coalitional actions against $S$, but play best responses individually according to Nash behavior. In addition, $S$ knows this and also chooses so as to maximize its own payoff. However, if finding the best response is difficult, which is often true in auctions with externalities, the concept of $\gamma$-core may become impractical. Especially in terms of fairness, it might be unreasonable that a bidder claims that an outcome is unfair even if IR by $\gamma$-effectivenesses does not hold since $\gamma$-effectiveness assumes other players play an equilibrium.

Beyond $\gamma$-effectiveness, in the auction context, it might also be reasonable to use subgame perfect equilibria for the two-stage extended game $\Gamma$, i.e., the first stage is the decision of participation, and the second stage is an auction, as in Jehiel and Moldovanu (1996). While fascinating, the same practical limitations apply, as in $\gamma$-effectiveness.

One simple yet reasonable choice might be to expect $S^c$ ($0 \notin S$) to have an efficient outcome for the subtype $T_{-S} \equiv T_{S^c}$. That is, $v_e(S) = \begin{cases} \max_{i \in S} \sum_{j \in S} t_{ij} & 0 \in S \\ \sum_{j \in S} t_{w_{-S},j} & 0 \notin S \end{cases}$, where $w_{-S} \in \arg\max_{i \in S^c} \sum_{j \in S^c} t_{ij}$, which I will refer to as the coalitional value function by *e-effectiveness* (*e* from "efficient").[24] Although the core is a one-step deviation concept, it might still be reasonable to assume that $S^c$ will have an efficient outcome among them, since otherwise they will deviate again. In this sense, the $e$-core may be more stable than the $\alpha$-core. In addition, $e$-effectiveness may be reasonable especially for efficient auction mechanisms, since the winning of $w_{-S}$ is the outcome when $S^c$ participates the auction. It is easy to see that the $e$-core can be a strict subset of the $\alpha$-core. Furthermore, Proposition 1 holds with the $e$-core, i.e., a stronger (positive) result.

**Proposition 8.** *Proposition 1 holds with the e-core.*

*Proof.* It is sufficient to show that GR for $S^c$ ($0 \in S$) holds. Due to NPR, we do not need to compare the payment in the payoff. Thus, it is sufficient to show that the new allocation cannot change favorably to $S^c$. For a contradiction, suppose it can, which means

$$\sum_{j \in S^c} t_{w_S,j} > \sum_{j \in S^c} t_{wj}, \tag{8}$$

where $w_S = \arg\max_{i \in S} \sum_{j \in S} t_{ij}$ (assuming a tie, if exists, is broken by some tie-breaking rule). However, the winning of $w_S \neq w$, i.e., efficiency among $S$, means

$$\sum_{j \in S} t_{w_S,j} > \sum_{j \in S} t_{wj}. \tag{9}$$

Combining (8) and (9) yields $\sum_{j \in N^0} t_{w_S,j} > \sum_{j \in N^0} t_{wj}$, which contradicts that $w$ is the efficient allocation for $N^0$. $\square$

While $e$-effectiveness is reasonable, it might not be justified for all cases. For instance, in inefficient auctions, a bidder would not decide to forgo participation, expecting an efficient allocation in her absence. Likewise, a bidder cannot reasonably claim that the auction outcome is unfair by comparing the realized payoff to the payoff she would receive if she had not participated and an efficient outcome had been realized.

[24] When there are multiple $w_{-S}$'s due to a tie, the same value is chosen for $v_e(S)$. Note that the $e$-core is unrelated to the so-called $\varepsilon$-core (or *epsilon* core, see Shubik and Wooders (1983); Wooders and Zame (1984); Kaneko and Wooders (1989), for instance), which is for approximate cores.

Another simple yet reasonable alternative especially for auction mechanisms might be to expect $S^c$ $(0 \notin S)$ to have an outcome by the mechanism for the subtype $T_{-S} \equiv T_{S^c}$.

**Definition 6.** For $S \subseteq N^0$, the *coalitional value function* by *$\delta$-effectiveness with respect to a mechanism* $\varphi = (x, \rho)$ is defined as

$$v_\delta(S) = v_\delta(S; T, \varphi) = \begin{cases} \max_{i \in S} \sum_{j \in S} t_{ij} & 0 \in S \\ \sum_{j \in S} t_{x(T_{-S}),j} & 0 \notin S \end{cases}.$$ [25]

One reason why "$v_\delta(S) = \sum_{j \in S} t_{x(T_{-S}),j}$ for $0 \notin S$" but "$v_\delta(S) = \max_{i \in S} \sum_{j \in S} t_{ij}$ for $0 \in S$" is reasonable is that the core only considers a one-step deviation. When $S$ $(0 \notin S)$ decides not to participate, $S^c$ would "deviate" from nonparticipation, i.e., $S^c$ would run an auction and participate. If this auction outcome is not efficient among $S^c$, then $S' \subseteq S^c$ might deviate once more, but this is the second-step deviation. Note that $e$-effectiveness is a special case of $\delta$-effectiveness when the mechanism is efficient. While $\delta$-effectiveness is simple yet reasonable, one major weakness might be the assumption that the subtype, $T_{-S}$, of $S^c$ does not change with respect to the original type, $T$. Thus, for $\delta$-effectiveness to be meaningful, a mechanism should have good incentive properties so that $S$ can expect $S^c$ to still submit $T_{-S}$ to the mechanism. In any case, however, $\delta$-effectiveness might be reasonable enough especially in terms of fairness.

# B The core with no seller deviations

We already showed that there always exists an outcome where the seller has no incentive to deviate (e.g., Proposition 1). In many auctions, however, the seller "cannot" deviate. In government auctions, the auctioneer (i.e., the government) "cannot" deviate after the auction ends even if they receive a better offer. Even in non-government auctions, many auctioneers (e.g., Sotheby's, Christie's, and sellers in eBay who care their seller ratings) care about their reputation (which may eventually affect their long-term profit). Disallowing the seller's deviations leads to a nonempty core with interesting (but not desirable) core payoff profiles.

**Definition 7.** *No seller deviations* (NSD) mean the seller cannot deviate, i.e., the seller cannot participate in any blocking coalition.[26]

[25] As in $v_\delta(S; T, \varphi)$, the core should also be defined with respect to $\varphi$, i.e., $Core(N^0, v, \varphi)$.

[26] To avoid a pathological core outcome (e.g., the seller has a net positive payment to bidders, i.e., negative revenue) due to NSD, we assume that the outcome is IR for the seller, i.e., $p \geq 0$, assuming the seller's "deviation" is allowed. While NSD may sound simple, $v_\alpha(S)$ in Equation (2) needs to change completely

As it seems a fairly strong restriction, NSD guarantees at least a nonempty core, which contains $\mathbf{0} = (0; 0, 0, ..., 0)$, i.e., no sale with no transfers, and this is the only core payoff profile with no sale. In addition, by Lemma 4, GR is satisfied for losers with nonpositive payment. Thus, we can have other core payoff profiles with a condition that prevents bribing the winner not to buy as follows.

**Proposition 9.** *With no seller deviations (NSD), the core is nonempty and always includes* $\mathbf{0} = (0; 0, 0, ..., 0)$*, i.e., no sale with no transfers. The core is* $\{\mathbf{0}, (p; t_{w1}-p_1, t_{w2}-p_2, ..., t_{ww}-p_w, ..., t_{wn}-p_n)\}$*, where* $w \in N$ *and* $\mathbf{p}$ *satisfy*

$$p \geq 0, \quad \text{(IR for the seller)}$$

$$p_j \leq 0, \forall j \in N \backslash w, \quad \text{(GR for losers)}$$

$$t_{ww} - p_w \geq -\sum_{j \in \mathcal{N}_w} (t_{wj} - p_j), \quad \text{(GR for bidders including the winner)} \tag{10}$$

*where* $\mathcal{N}_i = \{j \in N \backslash i : t_{ij} - p_j < 0\}$.

*Proof.* No sale case: Due to NSD, it is clear that $(0; 0, 0, ..., 0)$ is in the core. Now to show that this is the only core payoff profile, if any $u_i < 0$ for some $i \in N$, it is not IR for $i$ because $i$ need not worry about any reallocation due to no seller deviations. Thus, $u_i \geq 0$ for all $i \in N$. However, if $u_i > 0$ for some $i \in N$, then $u_0 < 0$, which violates IR for the seller.

Sale to $w$ for $t_{ww} \geq 0$ case: the item is sold to bidder $w$ with $p_w \geq 0$ and $p_j \leq 0$ for all $j \in N \backslash w$. Note that $p_j < 0$ is possible if $p \geq 0$ (i.e., IR for the seller) due to NSD. If $p_j > 0$, IR for $j$ does not hold due to no reallocation. Since $p_j \leq 0$, GR for losers holds by Lemma 4. Thus, the only possible deviation left is when GR for $S$ with $w \in S$ does not hold. There can be two subcases: if $p_w > t_{ww}$ ("pay too much"), IR for $w$ does not hold. Another subcase is $t_{ww} - p_w < -\sum_{j \in \mathcal{N}_w} (t_{wj} - p_j)$ ("suffer too much"). In this case, $\mathcal{N}_w$ bribes the winner not to buy. Thus, (10) should hold, which also requires $t_{ww} - p_w \geq 0$, i.e., IR for the winner. □

NSD guarantees a nonempty core but not necessarily desirable one though we can choose some "better" outcome depending on desired criteria. Besides "useless" no sale with no transfers, interestingly, a core outcome may contain a subsidy (a positive transfer to a bidder), which might not be acceptable to the seller. In addition, the outcome may not be efficient.

---

as follows: $v_\alpha(S) = \begin{cases} \left\{\sum_{j \in S} t_{ij} : i \in S\right\} & 0 \in S \\ 0 & 0 \notin S, w \in S \\ \sum_{j \in S} t_{wj} & 0 \notin S, w \notin S \end{cases}$. Note that the winner can be anyone including the seller. While bidders including a winner can secure 0 by payment refusals, losers still suffer externalities but which are now fixed (i.e., not the worst) due to no allocation change by NSD.

As shown in Lemma 2, the seller's incentive to deviate means the existence of justified envy, which is undesirable. Thus, even if the seller cannot deviate, it might be better to choose an outcome that is free of the seller's incentive to deviate, and those outcomes can always be achieved, as shown in Proposition 1.

# C Positive Externalities and Negative Valuations

Now we also allow positive externalities and negative valuations. All the results still hold with minor technical changes. Note that $t_{ij}$ denotes the externality (e.g., $t_{ij} < 0$: negative externality) imposed on bidder $j$ when bidder $i \neq j$ wins the item, and $t_{jj}$ denotes the bidder $j$'s own valuation (e.g., $t_{jj} > 0$: positive valuation) of the item. For simplicity of exposition, however, we normalize $t_{0j} = 0$ for all $j \in N^0$ without loss of generality.[27] That is, the seller does not impose any externalities on bidders, and has zero value on the item.

Each bidder, however, can impose different externalities, $t_{i0}$, on the seller, i.e., when $w \in N$ wins, the IR condition for the seller is now $t_{w0} + p \geq 0$ (instead of $p \geq 0$). Nonzero $t_{i0}$ is not only needed for generality but also useful in practice. First, as optimal auctions (e.g., Myerson 1981; Riley and Samuelson 1981) have bidder-specific reserve prices to increase revenue, the seller can use $t_{i0}$ as a reserve price for bidder $i$ in core-selecting mechanisms.[28] Even for the seller who is more interested in efficiency than revenue, $t_{i0}$ can be used. Normally, in auction theory, we only consider the welfare of the seller (e.g., the government) and bidders (e.g., companies). If the government, however, knows externalities to consumers, then she can use $t_{i0}$ to increase the welfare (e.g., to mitigate monopoly) also including consumers.

Due to positive exteranalities and negative valuations, the following needs some changes. Since they are trivial, we just restate these with necessary changes without proofs. First, the efficient sale assumption should change as follows:

**Lemma 8** (Lemma 1)**.** *If* $\max_{i \in N} \sum_{j \in N^0} t_{ij} < 0$ *with* $t_{jj} > -t_{j0}$ *for some* $j \in N$*, the core is empty.*

**Assumption 2** (Assumption 1)**.** *There exists an efficient sale, i.e.,* $\max_{i \in N} \sum_{j \in N^0} t_{ij} \geq 0$*, or* $\max_{j \in N}\{t_{jj} - t_{j0}\} \leq 0$.

Without positive externalities, the payment in NJE outcomes cannot be negative, as shown in Lemma 3. Note that, as explained in the previous paragraph of Proposition 1,

[27] As mentioned in footnote 9, this normalization needs positive externalities and negative valuations. Note also that we cannot further normalize $t_{i0} = 0$ for all $i \in N^0$ without loss of generality.

[28] The seller may also want to "pay" to herself (which will cancel out) when justified envy exists with $t_{w'0} - t_{w0} > 0$ in Definition 2. While the seller's payment is not a payment from a bidder, as the seller has a right not to sell below a reserve price in practice, the definition of justified envy still makes sense. Note also that $t_{i0}$ cannot work as a reserve price in the VCG since it is not core-selecting.

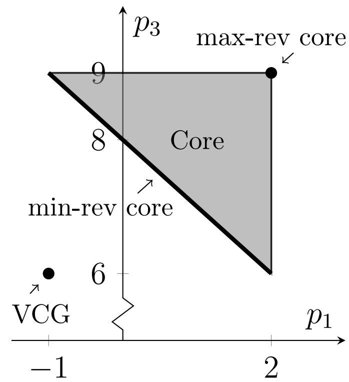


Figure 2: The core in Example 6

this lemma was introduced only to clearly explain one characteristic of the payments, i.e., $\mathbf{p} \geq 0$, which is implied by the condition (3) in the proposition. With positive externalities, however, the payment can be negative when the bidder imposes a positive externality on the seller as follows:

**Lemma 9** (Lemma 3)**.** *For an auction outcome* $(w, \mathbf{p})$ *to have no justified envy,* $t_{i0} + p_i \geq 0$ *for all* $i \in N$.

With these changes, all the results still hold. For instance, the core with NPR is still nonempty. We provide an example with positive externalities and negative valuations.

**Example 6.** For, $T = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \hline -1 & 2 & -7 & 9 \\ -1 & 0 & 1 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix}$. Note that now the seller is included, which is clearly shown by lines. The VCG outcome is $w = 1$, $p_1 = 0 - 1 = -1 < 0$ (i.e., a subsidy), $p_2 = 10 - 10 = 0$, and $p_3 = 0 - (-6) = 6$. The payoff profile $\mathbf{u} = (4; 3, -7, 3)$ is blocked by $\{0, 2\}$ since $t_{22} - (t_{12} - p_2) + t_{20} - t_{10} = b_{12} = 8 > 5 = p$, i.e., the VCG outcome is not in the core. For a core outcome while preventing the winner from being bribed by bidder 2, suppose we increase $p_3$ as much as possible, which is 9 by IR for bidder 3. Then, $p_1 \geq -1$ by $p \geq 8$, which leads to $u_1 = t_{11} - p_1 \leq 3$. But bidder 2 is willing to bribe bidder 1 not to buy by paying up to $-t_{12} = 7$, so the core is empty.

Figure 2 shows $(p_1, p_3)$ in the core with NPR (Proposition 1), which is $\{(-1 + p; 2 - p_1, -7, 9 - p_3) : p_1 + p_3 \geq 8, p_1 \leq 2, p_3 \leq 9\}$, a shaded region. The maximum-revenue core (Proposition 2) is $\{(10; 0, -7, 0)\}$, i.e., $w = 1$ and $p_N = (2, 0, 9)$. The minimum-revenue core is $\{(-1 + p; 2 - p_1, -7, 9 - p_3) : p_1 + p_3 = 8, p_1 \leq 2, p_3 \leq 9\}$, a thick line.

# References


Aumann, R. J. (1961). The core of a cooperative game without side payments. *Transactions of the American Mathematical Society 98*(3), 539–552.

Aumann, R. J. and B. Peleg (1960). Von Neumann-Morgenstern solutions to cooperative games without side payments. *Bulletin of the American Mathematical Society 66*(3), 173–179.

Ausubel, L. and P. Milgrom (2002). Ascending auctions with package bidding. *Frontiers of Theoretical Economics 1*(1), 1–42.

Ausubel, L. and P. Milgrom (2006). The lovely but lonely Vickrey auction. In *Combinatorial Auctions*, pp. 17–40. MIT Press.

Chander, P. and H. Tulkens (1997). The core of an economy with multilateral environmental externalities. *International Journal of Game Theory 26*(3), 379–401.

Chwe, M. S.-Y. (1994). Farsighted coalitional stability. *Journal of Economic Theory 63*(2), 299–325.

Clarke, E. H. (1971). Multipart pricing of public goods. *Public Choice 11*(1), 17–33.

Day, R. and P. Milgrom (2008). Core-selecting package auctions. *International Journal of Game Theory 36*(3-4), 393–407.

Day, R. and S. Raghavan (2007). Fair payments for efficient allocations in public sector combinatorial auctions. *Management Science 53*(9), 1389–1406.

Day, R. W. and P. Cramton (2012). Quadratic core-selecting payment rules for combinatorial auctions. *Operations Research 60*(3), 588–603.

Erdil, A. and P. Klemperer (2010). A new payment rule for core-selecting package auctions. *Journal of the European Economic Association 8*(2-3), 537–547.

Goeree, J. K. and Y. Lien (2016). On the impossibility of core-selecting auctions. *Theoretical economics 11*(1), 41–52.

Groves, T. (1973). Incentives in teams. *Econometrica 41*(4), 617–631.

Harsanyi, J. C. (1974). An equilibrium-point interpretation of stable sets and a proposed alternative definition. *Management science 20*(11), 1472–1495.

Hatfield, J. W. and P. R. Milgrom (2005). Matching with contracts. *The American Economic Review 95*(4), 913–935.

Jehiel, P. and B. Moldovanu (1996). Strategic nonparticipation. *RAND Journal of Economics 27*(1), 84–98.

Jehiel, P. and B. Moldovanu (1999). Resale markets and the assignment of property rights. *Review of Economic Studies 66*(4), 971–991.

Jehiel, P., B. Moldovanu, and E. Stacchetti (1996). How (not) to sell nuclear weapons. *American Economic Review 86*(4), 814–829.

Jeong, S. E. (2016). The Vickrey-Clarke-Groves with externalities: shill bidding that dominates truthful bidding. Working paper.

Jeong, S. E. (2017). Multidimensional second-price and English auctions. Working paper.

Kaneko, M. and M. H. Wooders (1989). The core of a continuum economy with widespread externalities and finite coalitions: from finite to continuum economies. *Journal of Economic Theory 49*(1), 135–168.

Kojima, F. (2012). The rural hospital theorem revisited. *International Journal of Economic Theory 8*(1), 67–76.

Milgrom, P. (2007). Package auctions and exchanges. *Econometrica 75*(4), 935–965.

Myerson, R. B. (1981). Optimal auction design. *Mathematics of Operations Research 6*(1), 58–73.

Ray, D. and R. Vohra (2015). The farsighted stable set. *Econometrica 83*(3), 977–1011.

Riley, J. G. and W. F. Samuelson (1981). Optimal auctions. *The American Economic Review 71*(3), 381–392.

Roth, A. E. (1986). On the allocation of residents to rural hospitals: a general property of two-sided matching markets. *Econometrica*, 425–427.

Shapley, L. and M. Shubik (1969). On the core of an economic system with externalities. *American Economic Review 59*(4), 678–684.

Shubik, M. and M. H. Wooders (1983). Approximate cores of replica games and economies. part i: Replica games, externalities, and approximate cores. *Mathematical Social Sciences 6*(1), 27–48.

Vickrey, W. (1961). Counterspeculation, auctions, and competitive sealed tenders. *Journal of Finance 16*(1), 8–37.

Wooders, M. H. and W. R. Zame (1984). Approximate cores of large games. *Econometrica*, 1327–1350.