Contests on Networks

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Abstract

We develop a model of contests on networks. Each player is "connected" to a set of contests, and exerts a single effort to increase the probability of winning each contest to which she is connected. We characterize equilibria under both the Tullock and all-pay auction contest success functions (CSFs), and show that many well-known results from the contest literature can be obtained by varying the structure of the network. We also obtain a new exclusion result: We show that, under both CSFs, equilibrium total effort may be higher when a strong player is excluded from the network. This finding contrasts the existing literature, which limits findings of this sort to the all-pay auction CSF. Our framework has a broad range of applications, including research and development, advertising, and research funding.

Keywords: Network Games, Contests, Bipartite Graph, Tullock Contest, All-pay Auction
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1 Introduction

In recent years, economists have recognized the importance of understanding how the structure of interactions affects economic behavior, which has led to the development of research on networks. The importance of this field of research is unquestionable, due to the broad applicability of these models in many economically relevant settings; to name a few: job search and employment dynamics (Calvó-Armengol and Jackson, 2004; Calvó-Armengol, 2004), the provision of public goods (Bramoullé and Kranton, 2007; Bramoullé et al., 2014), collaboration/research and development (Goyal and Moraga-Gonzalez, 2001; Goyal and Joshi, 2003), and criminal activity (Calvó-Armengol and Zenou, 2004; Ballester et al., 2006). Jackson and Zenou (2014) provide a comprehensive overview of the literature on network games, and emphasize three approaches researchers have taken to study such games:[1] (1) games of strategic complements and substitutes; (2) games with linear best-replies; and (3) settings with an uncertain pattern of interactions.[2] These three approaches have proved fruitful in allowing researchers to understand how the underlying pattern of interactions affects behavior.

In this paper, we study a new class of network games: contests on networks. Our model consists of set of players and a set of contests, which form a commonly known bipartite graph (or network).[3] Each player competes for prizes to which she is connected by exerting a single effort. A player's expected payoff depends on her own effort, and the efforts of her competitors. Contests can be used to model many economically relevant situations, and there is a vast literature that analyzes individual and aggregate behavior.[4] Our interest here is to understand how the pattern of interactions affects behavior. Our model has a number of interesting applications including, for example, centralized R&D decisions by multinational firms (MNFs). Prizes are commonly used tools to encourage R&D activity,[5] and contests can be used to model both explicit R&D contests or patent races (e.g. Che and Gale, 2003; Baye and Hoppe, 2003). To take advantage of economies of scale and scope, historically, R&D activity within MNFs has tended to be centralized, and undertaken at

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[1]See also Jackson (2008) and Bramoullé and Kranton (2016).

[2]Examples of models with an uncertain pattern of interactions include Jackson and Yariv (2007), Galeotti et al. (2010)

[3]A bipartite graph is a graph in which the vertices may be partitioned into two disjoint subsets, and the edges connect the vertices from these subsets.

[4]For relevant surveys see Nitzan (1994), Congleton et al. (2008), and Konrad (2009).

[5]Innovation prizes were a central feature of the Obama Administration's efforts to stimulate American innovation as part of the Recovery Act of 2009. From 2010 to 2012, 200 new innovation prizes were offered by federal agencies in areas ranging from national defense to education. (Source: `http://challenge.gov/about`.)

the corporate level (Gassmann and Von Zedtwitz, 1999). In this interpretation, the firm chooses a single level of R&D effort, the benefits of which are then realized by each branch of the firm. Our model could also be interpreted in the context of a national advertising campaign by a geographically dispersed franchised firm. In this context, each firm chooses a level of expenditure on a national advertising campaign, which increases the share of the market each franchise expects to capture. Finally, one might also interpret our model in the context of research funding. In this setting, researchers exert effort on a project proposal, which they then submit to various funding agencies to increase their chances of receiving funding for their project.

The main contributions of this research are twofold. First, we contribute to the literature on networks by analyzing a new class of network games. We characterize equilibrium behavior in terms of the underlying network characteristics, and study how these characteristics influence behavior. Second, we contribute to the literature on contests by establishing connections between several important observations in the contest literature, and different network structures in our setting. That is, we provide a unified framework, within which many well-known results in the contest literature can be obtained by simply varying the structure of the network. We also explore how equilibrium behavior depends on the contest success function (CSF). We focus on the two most widely used CSFs in the literature; the Tullock CSF, and the all-pay auction (APA) CSF.[6]

We show that the equilibrium behavior in symmetric contests[7] – a contest in which players compete for a single prize of common value – is analogous to equilibrium behavior (for both CSFs) in our model, when the network is biregular.[8] We also show that equilibrium behavior depends only on player and prize degrees, and is independent from the number of players and the number of prizes. Consistent with well-known results, when the network is biregular total equilibrium effort is always higher under the APA CSF than under the Tullock CSF.

We then show that equilibrium behavior in 2-player asymmetric contests[9] is akin to equilibrium behavior in our setting, when the network structure is a star.[10] In particular,

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[6]Comparisons between Tullock's CSF and the APA CSF are common in the literature. See, for example, Hillman and Riley (1989) or Fang (2002).

[7]See Tullock (1980) and Baye et al. (1996) for the characterization of equilibrium in symmetric contests with a single prize.

[8]A bipartite network, $\mathbf{g}$, is biregular if each pair of nodes in the disjoint subsets of $\mathbf{g}$ have the same degree.

[9]See Hillman and Riley (1989), Baye et al. (1996), Nti (1999, 2004), Stein (2002), and Matros (2006) for the characterization of equilibrium in asymmetric contests.

[10]A star network consists of a single "central player", $M \geq 1$ "periphery players" and M prizes. The central player competes for all M prizes, while each periphery player competes for exactly one prize.

the central player behaves *as if* she is competing in a two-player contest in which she has a higher value for the prize, while each periphery player behaves *as-if* competing in a two-player contest in which she has a lower value for the prize. For star networks under the Tullock CSF, total equilibrium effort is not monotonic in the level of noise in the CSF (i.e., Tullock's sensitivity parameter, r). This observation closely relates to Nti's (2004) result for two-player asymmetric contests. Moreover, under Tullock's CSF, the equilibrium effort of the central player is increasing in the number of prizes (and periphery players), while for the APA CSF, the central player's equilibrium strategy is independent from the number of prizes, and periphery players. Finally, we show that equilibrium effort may be higher under Tullock's CSF, than under the APA CSF.

One of the most striking observations in the contest literature is the Exclusion Principle, first introduced by Baye et al. (1993). The Exclusion Principle states that total equilibrium effort may increase if the most competitive (the highest value) player is excluded from the contest. It is a common finding in the contest literature that the Exclusion Principle holds only under the APA CSF, and does not apply to the Tullock CSF (see, for example, Fang, 2002; Matros, 2006; Menicucci, 2006). Intuitively, excluding the high-value player has two competing effects on equilibrium total effort. There is a direct effect: excluding this player decreases total effort, due to the loss of this his contribution. But there is also an indirect effect: the presence of a high-value player may have a "discouragement effect" on less competitive players. Excluding the high-value player "levels the playing field", which results in a more competitive contest, and leads the remaining players to exert higher effort. Prior results have found that the first effect always dominates the second under the Tullock CSF. The reason is that the Tullock CSF introduces a significant amount of noise in determining the outcome of the contest, as compared to the APA CSF. As a result, competition is softer, and the discouragement effect is less pronounced. In this paper, we derive a new exclusion principle, given in terms of network structures. Our result under the APA CSF nests the results of Baye et al., but we provide conditions on the network, under which our exclusion principle also applies to the Tullock CSF.

We are the first to study contests as a game on networks. Other studies of contests and networks have different angles. Franke and Öztürk (2015) study bilateral conflicts in which players may compete for several prizes and each player chooses a vector of efforts. König et al. (2015) studies a setting in which players, competing for a single prize, are linked as enemies or allies via a network structure. Kovenock and Roberson (2015) study the attack and defense of targets, which are connected via a network structure. In this model, the attacker's objective is to disconnect the network, while the defender's objective

is to maintain network connectivity. Marinucci and Vergote (2011) and Grandjean et al. (2016) study a model of network formation in an all-pay auction,[11] and a Tullock contest, respectively. In these models, players compete for a single prize, but the value of the prize to each player depends on the number of links she forms. Dahm and Esteve-Gonzalez (2014) and Dahm (2017) study contests on a particular network structure, in which all players compete for a main prize, while a set of disadvantaged players also compete for an additional prize. Both studies find that the additional prize can be used to level the playing field, and give some advantage back to the disadvantaged player(s). In this way, a contest designer may be able to elicit greater total effort by splitting the prize budget between two separate prizes.

The remainder of the paper is organized as follows. The model is presented in Section 2. In Section 3 we first characterize equilibria under both the Tullock and APA CSFs for certain network structures of interest. We then show the connection between our results and the existing contest literature. In Section 4 we study how the underlying network structure affects behavior. Concluding remarks are given in Section 5.

2   The Model

There are N players, and M contests. The set of players is denoted $\mathcal{N} = \{1,\ldots,N\}$; the set of contests is denoted $\mathcal{M} = \{1,\ldots,M\}$. Each player i is risk neutral and is characterized by a vector $\mathbf{g}_i \equiv (g_{i1},\ldots,g_{iM})$ where $g_{im} = 1$, if player i competes in contest m, and $g_{im} = 0$ otherwise. Let $d_i^p = \sum_{m \in \mathcal{M}} g_{im}$ denote the degree of player i; i.e., d_i^p is the number of contests in which i competes. Let $d_m^c = \sum_{i \in \mathcal{N}} g_{im}$ denote the degree of contest m; i.e., d_m^c is the number of players that compete in contest m. We assume throughout that for all $m \in \mathcal{M}$, $d_m^c \geq 2$, which ensures that there is at least some competition in each contest. Associated with each contest is a prize; if player i wins contest m, then she receives the prize, $V > 0$. Player i chooses a single effort, $x_i \in \mathbb{R}_+$, to increase her probability of winning each contest in which she competes.

The network structure can be represented by a bipartite graph - a graph in which the vertices can be separated into two disjoint subsets, and each edge connects the vertices from these subsets. In our setting, the two disjoint subsets are the set of players, $\mathcal{N}$, and the set of contests, $\mathcal{M}$; the edges indicate in which contest(s) each player competes.

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[11]Marinucci and Vergote consider an all-pay auction where players have private values. In contrast, we study the complete information all-pay auction. These two models are quite different qualitatively; in particular, in contrast to the private value case, no pure strategy equilibrium exists under complete information.
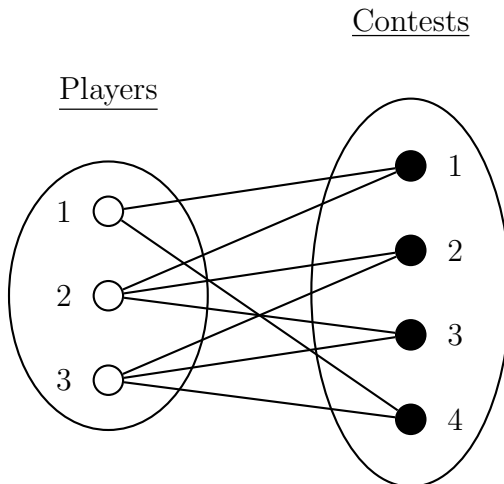
Figure 1: The contest network as a bipartite graph.

Figure 1 illustrates the bipartite structure of the network. In this figure, player 1 competes in two contests ($d_1^p = 2$); players 2 and 3 each compete in three contests ($d_2^p = d_3^p = 3$); and 2 players compete in each contest ($d_m^c = 2$ for each contest $m = 1, 2, 3, 4$). The bipartite network structure is summarized by the $N \times M$ biadjacency matrix:

$$\mathbf{g} = \begin{pmatrix} g_{11} & g_{12} & \cdots & g_{1M} \\ g_{21} & g_{22} & \cdots & g_{2M} \\ \vdots & \vdots & \ddots & \vdots \\ g_{N1} & g_{N2} & \cdots & g_{NM} \end{pmatrix}.$$

For the network structure in Figure 1:

$$\mathbf{g} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}.$$

Denote by $\mathbf{x} = (x_1, \ldots, x_N)$ the vector of all players' efforts. For a given network structure, $\mathbf{g}$, and vector of efforts, $\mathbf{x}$, let $p_{im}(\mathbf{x}, \mathbf{g})$ denote the probability that player i wins contest m. The function, $p_{im}(\cdot)$ is the contest-success function (CSF). Note that $p_{im}(\mathbf{x}, \mathbf{g}) = \mathbf{0}$ if $g_{im} = 0$. Moreover, we assume that p_{im} depends only on the efforts allocated to contest m, and does not depend on the outcomes of other contests.

The expected payoff to player i is the sum of the expected payoffs across contests in

which she competes; it is given by,

$$\pi_i(\mathbf{x}, \mathbf{g}) = \sum_{m \in \mathcal{M}} p_{im}(\mathbf{x}, \mathbf{g})V - x_i. \tag{1}$$

Player i takes the strategies of the other $N-1$ players and the network structure as given, and chooses x_i to maximize π_i. Before proceeding to the analysis, there are a few network structures worth mentioning.

Biregular Networks

A bipartite network, $\mathbf{g}$, is *biregular* if each pair of nodes in the disjoint subsets of $\mathbf{g}$ have the same degree. In our context, this means each player competes in the same number of contests, and each contest has the same number of participants. We provide the following definition:

Definition 1. *Biregular Network*

The bipartite network, $\mathbf{g}$, is biregular if for all contests $m, n \in \mathcal{M}$, and all players $i, j \in \mathcal{N}$:

1. *Contest Symmetry: $d^c_m = d^c_n = d^c$*

2. *Player Symmetry: $d^p_i = d^p_j = d^p$*

In terms of the biadjacency matrix, for a biregular network the sum across each row is equal to d^p, and the sum down each column is equal to d^c. Studying Figure 1, it is clear that this is not a biregular network; while each contest has 2 participants (so Part 1 of Definition 1 is satisfied, with $d^c = 2$), player 1 competes in 2 contests, while players 2 and 3 each compete in 3 contests; thus, Part 2 of Definition 1 is not satisfied.

A biregular network with N players, M prizes, contest degree, d^c, and player degree, d^p, can be summarized by $[N, d^p; M, d^c]$. From the degree definitions (and our assumption that $d^c_m \geq 2$) it follows,

$$2 \leq d^c \leq N \tag{2}$$

and

$$1 \leq d^p \leq M. \tag{3}$$

Note that the following *link property* must hold for biregular networks:

$$N d^p = M d^c. \tag{4}$$

The left-hand side of (4) gives the number of links from players to prizes. The right-hand side of (4) gives the number of links from prizes to players. Clearly, these numbers must be the same. Two special cases of biregular networks are *complete networks*, and *circle networks*.[12] We provide the following definitions:

Definition 2. *Complete Network*

    A biregular network is complete if $d^c = N$.

Definition 2 says that the biregular network is complete if each contest has N participants. Note that the link property (4) implies that in a complete network, each player competes for all M prizes; i.e., $d^p = M$. This means that there is always a unique complete network for any given N and M. We next define a circle network. Before doing so, we will need to introduce some additional terminology: A *walk* is a sequence of nodes, $\ell_1, \ldots, \ell_k$, where $\ell_i \in \mathcal{N} \cup \mathcal{M}$, and for each i from 1 to $k-1$, ℓ_i is linked to ℓ_{i+1}.[13] The length of the walk is equal to the number of links; i.e., $k-1$. A *path* is a walk $\ell_1, \ldots, \ell_k$ such that each pair of nodes is distinct, $\ell_i \neq \ell_j$, with the possible exception that $\ell_1 = \ell_k$. A *cycle* $\ell_1, \ldots, \ell_k$ is a path such that $\ell_1 = \ell_k$.

Definition 3. *Circle Network*

    A biregular network is a circle if $d^c = d^p = 2$, and there is a cycle of length $2N$.

In a circle network, each player competes in two contests, and each contest has two participants (i.e., $d^p = d^c = 2$), but no two contests have the same set of participants. Note that the link property, (4), then implies $N = M$. Example 1 describes all biregular networks for the case of $N = 3$; Figure 2 illustrates the corresponding networks.

Example 1. *Suppose that there are $N = 3$ players.*

- *If $M = 1$, then there exists a unique biregular network. In this network, $d^c = 3$ and $d^p = 1$.*

- *If $M = 2$, then there exists a unique biregular network. In this network, $d^c = 3$ and $d^p = 2$.*

- *If $M = 3$, then there exist two biregular networks:*

    1. *A circle network where $d^c = d^p = 2$.*
    2. *A complete network where $d^c = d^p = 3$.*

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[12]Circle networks are also known as cycle graphs or ring networks.

[13]For instance, if $\ell_1 \in \mathcal{N}$, and k is odd then it must be that $\ell_k \in \mathcal{N}$. If there is a walk from ℓ_1 to ℓ_k then there exist nodes $\ell_2, \ldots, \ell_{k-1} \in \mathcal{M} \cup \mathcal{N}$ such that $g_{\ell_1 \ell_2} = g_{\ell_3 \ell_2} = g_{\ell_3 \ell_4} = \cdots = g_{\ell_k \ell_{k-1}} = 1$.
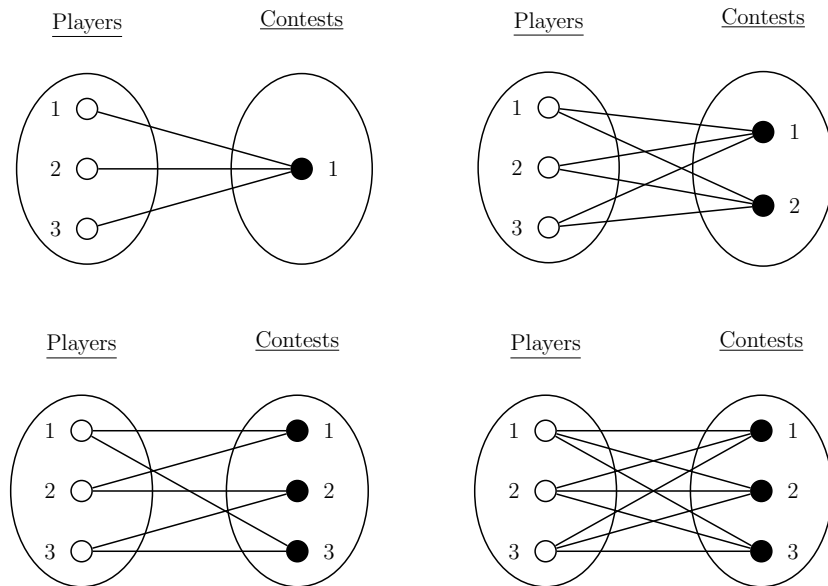
Figure 2: Illustration of the four networks described in Example 1. For all four networks, $N = 3$. Top left: $M = 1$, $d^c = 3$, $d^p = 1$; Top right: $M = d^p = 2$, $d^c = 3$; Bottom left: A circle network with $M = 3$, $d^c = d^p = 2$; Bottom right: A complete network with $M = d^c = d^p = 3$.

Star Networks

In a *star network* there are M contests and $N = M + 1$ players: M "periphery players" and 1 "central player". Each periphery player competes in a single contest, while the central player competes in all M contests. The winner of each contest receives a prize, $V > 0$.

Definition 4. *Star Network*

    Let **g** *be a network such that* $|\mathcal{M}| = M \geq 2$ *and* $|\mathcal{N}| = M + 1$. *The network is a star if (1) each contest* $k \in \mathcal{M}$ *has degree 2, i.e.,* $d^c_k = 2$; *(2) there is a "central player"* $i^* \in \mathcal{N}$ *that has degree* M, *i.e.,* $d^p_{i^*} = M$; *and (3) each "periphery player"* $j \in \mathcal{N} \setminus \{i^*\}$ *has degree 1, i.e.,* $d^p_j = 1$.

Figure 3 illustrates a star network where $M = 5$. As a convention, we will label the central player of a star network as player $M + 1$; so, the set of periphery players is $\{1, \ldots, M\}$.

Hybrid Networks

Finally, we describe *hybrid networks*. Let **g** be a biregular network, and, consider adding an additional player, who competes in σ contests. We refer to the resulting network
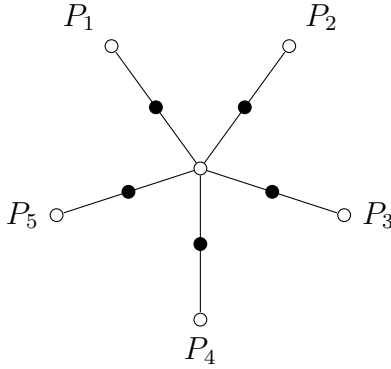
Figure 3: A star network with $M = 5$. Players are represented by hollow nodes; contests are represented by solid nodes.

structure as a σ-hybrid network.[14] We provide the following definition:

Definition 5. *σ-Hybrid Network*

Let $\mathbf{g}$ be any biregular network with N players and M prizes. A σ-hybrid network is the network, $\mathbf{g}'$, formed by adding an additional player to $\mathbf{g}$, who competes for $\sigma \leq M$ prizes. We call $\mathbf{g}$ the underlying network of $\mathbf{g}'$. We refer to the players in $\mathbf{g}$, $\{1, \ldots, N\}$, as the underlying players of $\mathbf{g}'$, and we call the additional player, $N + 1$, the hybrid player of $\mathbf{g}'$.

Figure 4 illustrates a 6-hybrid network where the underlying network is a 6-player circle.

3   A Unified Framework

In this section we show that a number of important results from the contest literature can be obtained as special cases of our network framework. Section 3.1 sets up the model under the Tullock CSF, and provides an explicit characterization of equilibrium for certain network structures of interest. Section 3.2 provides an analogous analysis for the APA CSF. Section 3.3 unifies our results with the existing literature on contests. All proofs are contained in the Appendix.

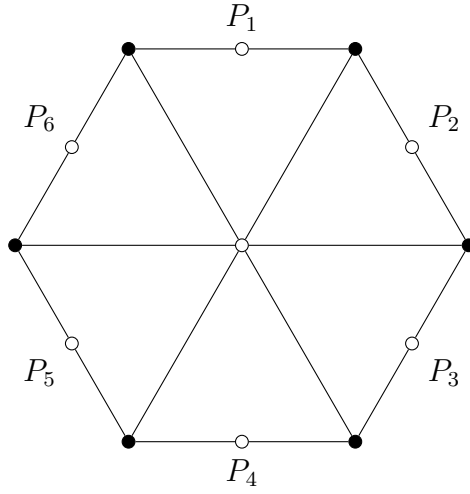[14]We will sometimes refer to a σ-hybrid simply as a hybrid.

Figure 4: A 6-hybrid network. The underlying network is a 6-player circle. Players are represented by hollow nodes; contests are represented by solid nodes.

3.1 Tullock CSF

We now extend the classical Tullock contest to our network contest environment. Fix a network structure, $\mathbf{g}$, and let $\mathbf{x}_{-i} \in \mathbb{R}_+^{N-1}$ denote the vector of efforts chosen by all players other than some player i. If $\sum_{j \in \mathcal{N}} g_{jm} x_j > 0$ for some prize m, then the probability that player i wins prize m is given by the following contest success function:

$$p_{im}(x_i, \mathbf{x}_{-i}, \mathbf{g}) = \frac{g_{im} x_i^r}{\sum_{j \in \mathcal{N}} g_{jm} x_j^r}.$$

The parameter $r > 0$ measures the sensitivity of the probability of success to players' effort choices. A higher value of r corresponds to a contest success function that is more sensitive to players' efforts. If $\sum_{j \in \mathcal{N}} g_{jm} x_j = 0$ for some contest m, then we assume that each player linked to m is equally likely to win: $p_{im}(\cdot) = \frac{g_{im}}{d_m^c}$. Using equation (1), the expected payoff to player i may then be written:

$$\pi_i(x_i, \mathbf{x}_{-i}, \mathbf{g}) = \sum_{m \in \mathcal{M}} \frac{g_{im} x_i^r}{\sum_{j \in \mathcal{N}} g_{jm} x_j^r} V - x_i.$$

Player i takes $\mathbf{x}_{-i}$ and $\mathbf{g}$ as given and chooses $x_i \geq 0$ to maximize $\pi_i(x_i, \mathbf{x}_{-i}, \mathbf{g})$. At

an interior solution, the first-order condition for player i is,

$$\frac{\partial \pi_i(x_i, \mathbf{x}_{-i}, \mathbf{g})}{\partial x_i} = \sum_{m \in \mathcal{M}} \left\{ \frac{rx_i^{r-1}\left(\sum_{j \in \mathcal{N}} g_{jm}x_j^r\right) - rx_i^{2r-1}}{\left(\sum_{j \in \mathcal{N}} g_{jm}x_j^r\right)^2} g_{im}V \right\} - 1 = 0. \qquad (5)$$

In a pure-strategy equilibrium in which each player is active (i.e. chooses a strictly positive effort level), equation (5) is satisfied for each player i.

Characterizing Equilibrium

Equation (5) characterizes equilibrium for general network structures (when all players are active). We will now provide closed-form expressions for equilibrium efforts for certain network structures of interest. It will be seen in Section 3.3 that these networks have interesting connections with the existing contest literature. We will focus on "symmetric" equilibria. By symmetric in this context, we mean that symmetric players follow symmetric strategies.

To begin, suppose that the network is biregular, summarized by $[N, d^p; M, d^c]$, where $d^c \geq 2$. In a biregular network, all players are symmetric; so, in a symmetric equilibrium, $x_i^* = x_j^* = x^*$ for each $i, j \in \mathcal{N}$. Imposing symmetry, for each $i \in \mathcal{N}$, equation (5) simplifies to,

$$\sum_{m \in \mathcal{M}} \left\{ \frac{rx^{*r-1}\left(\sum_{j \in \mathcal{N}} g_{jm}x^{*r}\right) - rx^{*2r-1}}{\left(\sum_{j \in \mathcal{N}} g_{jm}x^{*r}\right)^2} g_{im}V \right\} - 1 = 0. \qquad (6)$$

Applying the properties of biregular networks given in Definition 1 to equation (6), our first result characterizes the unique symmetric equilibrium of biregular networks.

Lemma 1. *Suppose that the network is biregular and summarized by $[N, d^p; M, d^c]$. Under the Tullock CSF there exists a symmetric pure-strategy equilibrium if and only if $r \leq \frac{d^c}{d^c-1}$. The symmetric equilibrium is unique, and equilibrium individual effort, x^*, is given by,*

$$x^* = \frac{r(d^c - 1)}{(d^c)^2} d^p V.$$

Total equilibrium effort is,

$$X^* = Nx^* = \frac{r(d^c - 1)}{d^c}MV.$$

Next, consider a star with M contests. Suppose that each of the M symmetric periphery players follow symmetric strategies. Let x^* denote the equilibrium effort of each periphery player, and let x^*_{M+1} denote the equilibrium effort of the central player. For each periphery player, (5) reduces to,

$$\frac{rx^{*r}x^{*r-1}_{M+1}}{(x^{*r} + x^{*r}_{M+1})^2}V = 1. \tag{7}$$

While for the central player, (5) becomes,

$$\sum_{i=1}^{M} \frac{rx^{*r}x^{*r-1}_{M+1}}{(x^{*r} + x^{*r}_{M+1})^2}V - 1 = 0. \tag{8}$$

Our next result characterizes the unique symmetric equilibrium when the network is a star.

Lemma 2. *Suppose the network is a star with M contests. Under the Tullock CSF there exists a symmetric pure-strategy equilibrium if and only if $rM^r \leq M^r + 1$. The symmetric equilibrium is unique, and effort for each periphery player is,*

$$x^* = \frac{rM^r}{(M^r + 1)^2}V.$$

The effort of the central player is $x^*_{M+1} = Mx^*$. Total equilibrium effort is,

$$X^* = Mx^* + x^*_{M+1} = \frac{2rM^{r+1}}{(M^r + 1)^2}V. \tag{9}$$

Finally, consider a σ-hybrid network. In contrast to biregular and star networks, for hybrid networks, there may not exist a symmetric equilibrium in which every player is active (i.e. exerts strictly positive effort). In particular, unless σ is large enough, there will always exist a symmetric equilibrium in which the hybrid player is inactive (i.e., exerts zero effort). We illustrate this finding first via an example, and then we provide a general result.

Example 2. *Suppose the network is a 1-hybrid in which the underlying biregular network is a 3-player circle. The network structure is illustrated in Figure 5. Note that in any*

symmetric equilibrium, $x_1^* = x_3^*$. Under the Tullock CSF with $r = 1$, it may be verified that there is only one symmetric equilibrium, and in this equilibrium, $x_1^* = x_2^* = x_3^* = \frac{V}{2}$; and $x_4^* = 0$.
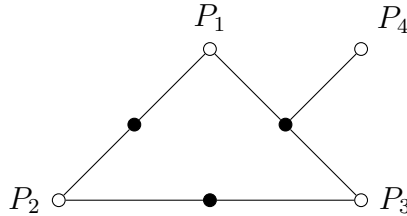

Figure 5: A 1-hybrid network in which the underlying biregular network is a 3-player circle. Players are represented by hollow nodes; contests are represented by solid nodes.

Our next result generalizes the finding of Example 2, and shows that, in a 1-hybrid, there always exists an equilibrium in which the hybrid player is inactive, so long as each of the other players compete in at least 2 contests.

Lemma 3. *Consider a 1-hybrid network, $\mathbf{g}'$, with the underlying network, $\mathbf{g}$, where $\mathbf{g}$ is summarized by $[N, d^p; M, d^c]$ with $d^p \geq 2$. Under the Tullock CSF ($r = 1$) there exists an equilibrium in which the hybrid player is inactive, and the effort of each underlying player is given by,*

$$x^* = \frac{(d^c - 1)}{(d^c)^2} d^p V.$$

Example 2 and Lemma 3 suggest that when an additional player is added to a symmetric network, that player must compete in several contests in order to have an incentive to actively participate. We will next consider the extreme case of M-hybrid networks, in which the hybrid player competes for all M prizes in the underlying network. Figure 4 illustrates an M-hybrid network in which the underlying network is a 6-player circle.

Lemma 4. *Suppose the network is an M-hybrid with an underlying biregular network summarized by $[N, d^p; M, d^c]$. Under the Tullock CSF with $r = 1$ there exists a unique symmetric equilibrium. In this equilibrium, effort for each underlying player is,*

$$x^* = \frac{N}{(N + 1)^2} d^p V.$$

The effort of the hybrid player is,

$$x_{N+1}^* = (N + 1 - d^c)x^*.$$

Total equilibrium efforts is,

$$X^* = Nx^* + x^*_{N+1} = \frac{(2N+1-d^c)N}{(N+1)^2} d^p V.$$

3.2 APA CSF

In this section we extend the all-pay auction to our network framework. Under the all-pay auction CSF, player i wins contest m with certainty if her effort choice is greater than the effort exerted by the other participants of contest m. In the event that 2 or more players choose the same highest effort, these players are equally likely to win. Formally:

$$p_{im}(x_i, \mathbf{x}_{-i}, \mathbf{g}) = \begin{cases} g_{im}, & \text{if } x_i > \max_{j \neq i}\{g_{jm}x_j\}, \\ 0, & \text{if } x_i < \max_{j \neq i}\{g_{jm}x_j\}, \\ \frac{g_{im}}{n}, & \text{if } i \text{ ties with } n-1 \text{ others for the highest bid.} \end{cases}$$

It is well-known that the APA with complete information does not possess a pure-strategy equilibrium. Denote by F_j the CDF of player j's mixed strategy, and let $\mathbf{F}_{-i} = (F_1, \ldots, F_{i-1}, F_{i+1}, \ldots, F_N)$ denote the collection of (independent) distribution functions of all players other than player i. If player i chooses effort x_i then,

$$p_{im}(x_i, \mathbf{F}_{-i}, \mathbf{g}) = g_{im} \prod_{j \neq i} F_j(x_i)^{g_{jm}}.$$

The expected payoff for player i is,

$$\pi_i(x_i, \mathbf{F}_{-i}, \mathbf{g}) = \sum_{m \in \mathcal{M}} \left(g_{im} \prod_{j \neq i} F_j(x_i)^{g_{jm}} \right) V - x_i.$$

Let $S_i \subseteq \mathbb{R}_+$ denote the support of player i's mixed strategy. In equilibrium, for each player i, and each $x, x' \in \mathbb{R}_+$,

$$x, x' \in S_i \implies \pi_i(x, \mathbf{F}_{-i}, \mathbf{g}) = \pi_i(x', \mathbf{F}_{-i}, \mathbf{g})$$

and

$$x \in S_i, x' \notin S_i \implies \pi_i(x, \mathbf{F}_{-i}, \mathbf{g}) \geq \pi_i(x', \mathbf{F}_{-i}, \mathbf{g})$$

Characterizing Equilibrium

We now provide an explicit characterization of equilibrium distribution functions for the network structures of interest. Before proceeding, note that in a single-prize contest under the APA CSF there typically exist many equilibria, including a continuum of asymmetric equilibria when all players have a common prize value (see Baye et al., 1996, henceforth, BKD). However, there typically exists a unique symmetric equilibrium. In our analysis, we will focus on "symmetric" equilibria. As in Section 3.1, when we say "symmetric" in this context, we mean that symmetric players follow symmetric strategies. We now characterize equilibrium for biregular networks under the APA CSF.

Lemma 5. *Suppose the network is biregular. Under the APA CSF there is a unique symmetric equilibrium in which each player randomizes continuously on the interval $[0, d^p V]$ according to the CDF,*

$$F^*(x) = \left(\frac{x}{d^p V} \right)^{\frac{1}{d^c - 1}}. \tag{10}$$

The equilibrium expected payoff to each player is zero. The expected total equilibrium effort is

$$X^* = \frac{N d^p}{d^c} V = M V.$$

Our next result characterizes the unique symmetric equilibrium for a star network.

Lemma 6. *Suppose the network is a star with M contests. Under the APA CSF there exists a unique symmetric equilibrium in which the central player randomizes uniformly on $[0, V]$, while each periphery player places an atom at 0 of size $\alpha = \frac{M-1}{M}$, and randomizes continuously on $(0, V]$ according to the CDF,*

$$F^*(x) = \frac{M-1}{M} + \frac{x}{MV}.$$

The expected payoff of the central player is $(M-1)V$. The expected payoff of each periphery player is zero, and the expected total equilibrium effort is,

$$X^* = V.$$

Next, we consider hybrid networks under the APA CSF. Our first result complements the finding of Lemma 3, and shows that unless the hybrid player competes in sufficiently many contests, there always exists a symmetric equilibrium in which she is inactive.

Lemma 7. *Consider a 1-hybrid network, $\mathbf{g}'$, with an underlying network, $\mathbf{g}$, where $\mathbf{g}$ is summarized by $[N, d^p; M, d^c]$ with $d^p \geq 2$. Under the APA CSF there exists an equilibrium in which the hybrid player is inactive, and each underlying player randomizes continuously on $[0, d^p V]$ according to the CDF,*

$$F^*(x) = \left(\frac{x}{d^p V} \right)^{\frac{1}{d^c - 1}}.$$

Finally, we characterize equilibrium under the APA CSF when the hybrid player competes for all M of the contests.

Lemma 8. *Suppose the network is an M-hybrid with an underlying network summarized by $[N, d^p; M, d^c]$. Under the APA CSF there exists a unique symmetric equilibrium in which each underlying player places an atom of size $\alpha = \left(\frac{M - d^p}{M} \right)^{\frac{1}{d^c}}$ at zero, and randomizes continuously on $(0, d^p V]$ according to the CDF,*

$$F^*(x) = \left(\frac{M - d^p}{M} + \frac{x}{MV} \right)^{\frac{1}{d^c}}.$$

The hybrid player randomizes continuously on $[0, d^p V]$ according to the CDF,

$$G^*(x) = \frac{x}{d^p V F(x)^{d^c - 1}}.$$

The expected payoff of the hybrid player is $(M - d^p)V$. The expected payoff of each underlying player is zero, and the expected total effort is equal to

$$X^* = d^p V.$$

3.3   Unification

In this section, we unify the results of Sections 3.1-3.2 with the existing literature on contests.

Symmetric Contests

We begin by showing a connection between behavior in single-prize symmetric contests and behavior in our model when the network structure is biregular.

Theorem 1. *Suppose that the network is biregular and the CSF is either the Tullock or APA. Individual equilibrium behavior and expected payoffs are equivalent to the behavior*

and expected payoffs in a d^c-player contest in which symmetric players compete for a single prize with common value d^pV. Total equilibrium effort is equivalent to total equilibrium effort in a d^c-player contest in which players compete for a single prize with common value MV.

Theorem 1 establishes a connection between behavior in the contest played on the network, and behavior in symmetric single-prize contests, absent network effects. Consider a biregular network summarized by $[N, d^p; M, d^c]$; in this network each of the N players competes for d^p prizes, and each contest has d^c participants. Each player thus competes for a total value of d^pV, while the total value of all prizes is MV. Theorem 1 establishes that each player behaves "as if" competing in a d^c-player contest in which each player has common value d^pV, while total equilibrium effort is "as if" d^c players each compete for a prize of common value MV.

To further illustrate this result, consider, for concreteness, the three-player circle illustrated in the bottom-left panel of Figure 2. In this network, each player competes in two contests, and each contest has two participants. In equilibrium, each player behaves as if competing for a single prize of size $2V$ against one other player. To see this connection, first consider a two-player contest with players A and B, who each compete for a single prize of size $2V$, and suppose the CSF is the Tullock. Let x^* denote the (unique) symmetric equilibrium effort level in this game. If player A chooses effort, x_A, and B plays according to equilibrium, $x_B = x^*$, then the payoff to player A is:

$$\pi_A(x_A, x_B = x^*) = \frac{x_A^r}{x_A^r + x_B^r}2V - x_A = \frac{x_A^r}{x_A^r + x^{*r}}V + \frac{x_A^r}{x_A^r + x^{*r}}V - x_A. \qquad (11)$$

Now, in the 3-player circle network, if Player 1 (say) chooses effort x_1, while players 2 and 3 each choose effort, x^*, then the payoff to Player 1 is the sum of the payoffs she receives from the contest with Player 2 and with Player 3:

$$\pi_1(x_1, x_2 = x^*, x_3 = x^*, \mathbf{g}) = \frac{x_1^r}{x_1^r + x_2^r}V + \frac{x_1^r}{x_1^r + x_3^r}V - x_1$$

$$= \frac{x_1^r}{x_1^r + x^{*r}}V + \frac{x_1^r}{x_1^r + x^{*r}}V - x_1. \qquad (12)$$

Clearly, the payoff to player A in equation (11) is the same as the payoff to player 1 given in equation (12). Since, by definition, x^* maximizes the payoff to A, it must also

maximize Player 1's payoff. Hence, when Players 2 and 3 choose effort x^*, Player 1's best response is also to choose x^*.[15] An analogous symmetry holds under the APA CSF.

It is also worth mentioning that the standard comparative statics results from the contest literature hold in our setting when the network is biregular. Specifically, for either the APA or Tullock CSF, expected individual and total equilibrium effort are increasing in the prize value V. In addition, under the Tullock CSF, individual and total equilibrium efforts are increasing in the noise parameter, r, and for any r such that a pure strategy equilibrium exists under the Tullock, expected total effort is higher under the APA than under the Tullock CSF.

Asymmetric Contests

Our next result establishes a connection between equilibrium behavior in the star network with equilibrium behavior in 2-player asymmetric contests.

Theorem 2. *Suppose the network is a star and the CSF is either the Tullock or APA. Individual equilibrium behavior and expected payoffs are equivalent to the behavior and expected payoffs in a 2-player asymmetric contest in which the "weak" player has value, V, while the "strong" player has value, MV. Each periphery player behaves as-if competing as the weak player, while the central player behaves as-if competing as the strong player in the 2-player contest.*

Theorem 2 establishes a close connection between a contest played on a star network and 2-player asymmetric contests. To see this connection more explicitly, consider a star network with M contests. Note that the payoff to a periphery player, i, only depends directly on her own effort and the effort of the central player. So, let $\pi_i(x, x_{M+1})$ denote the payoff to player i when she chooses effort x, and the central player, $M + 1$, chooses effort x_{M+1}. Under the Tullock CSF,

$$\pi_i(x, x_{M+1}) = \frac{x^r}{x^r + x_{M+1}^r} V - x. \tag{13}$$

The expected payoff to the central player depends on her own effort as well as the efforts of each periphery player. Suppose the periphery players choose efforts according to $\mathbf{x} = (x_1, \ldots, x_M)$. Then, the central player's expected payoff, $\pi_{M+1}(x_{M+1}, \mathbf{x})$, is given by,

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[15]Given the symmetric nature of the network, our choice to examine the payoff of Player 1 was arbitrary; this same analysis could be applied for players 2, or 3.

$$\pi_{M+1}(x_{M+1}, \mathbf{x}) = \sum_{i=1}^{M} \frac{x_{M+1}^r}{x_i^r + x_{M+1}^r} V - x_{M+1}.$$

If the periphery players follow symmetric strategies, so that $\mathbf{x} = (x, \ldots, x)$, then the expected payoff to the central player reduces to,

$$\pi_{M+1}(x_{M+1}, \mathbf{x}) = \frac{x_{M+1}^r}{x_{M+1}^r + x^r} MV - x_{M+1}. \tag{14}$$

Studying expressions (13) and (14) it is clear that, when the periphery players adopt symmetric strategies, the expected payoff of the central player is equivalent to what his payoff would be in a two-player Tullock contest in which he has value MV and his opponent chooses effort, x. Moreover, the expected payoff to each periphery player is equivalent to what her payoff would be in a two-player contest in which she has value V. A similar symmetry holds under the APA CSF. Note, however, that total equilibrium effort on the star is greater than it would be in a 2-player asymmetric contest. This is clearly the case since, on the star network, there are $M \geq 1$ periphery players, each of whom behaves as they would in the 2-player contest.

As shown in Section 3.1, for biregular networks under the Tullock CSF total equilibrium effort is linear and increasing in r. This result is consistent with findings in the contest literature when players have the same prize values. When the prize values are different, Nti (1999, 2004) shows that this relationship is less clear, and is generally not monotonic. We obtain a similar finding in our setting when the network is a star.

In what follows, we restrict attention to values of r that permit the existence of a symmetric pure-strategy equilibrium. Let $\bar{r}_M$ denote the unique solution to the equation

$$(\bar{r}_M - 1)M^{\bar{r}_M} = 1. \tag{15}$$

$\bar{r}_M$ is the highest value of r that supports a symmetric pure-strategy equilibrium on a star with M periphery players (see Lemma 2). Clearly, for any M, $\bar{r}_M > 1$. Moreover, it is straightforward to show that $\bar{r}_M$ is strictly decreasing in M, and $\bar{r}_M \in (1, 2]$.[16]

Let $X_M^*(r)$ denote total equilibrium effort when there are M periphery players, and the Tullock CSF parameter is r. Let $r^*(M)$ denote the value of r that maximizes total

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[16]It is well-known that for $r \in (2, \infty)$ there does not exist a pure-strategy equilibrium under the Tullock CSF. For such values of r, the mixed-strategy equilibrium shares some common features with equilibrium under the APA CSF. See, Baye et al. (1994), Alcalde and Dahm (2010), and Ewerhart (2015).

equilibrium effort:

$$r^*(M) = \arg \max_{r \in [0, \overline{r}_M]} \{X_M^*(r)\}.$$

Our next proposition reveals that total equilibrium effort is in fact a single-peaked function of r. Moreover, we show that for any fixed $r \in (0, \overline{r}_M)$, total equilibrium effort is increasing in M. That is, as the number of periphery players (and prizes) increases, total equilibrium effort increases.

Proposition 1. *Suppose the network is a star with $M \geq 2$ contests, and the CSF is the Tullock. Equilibrium total effort, $X_M^*(\cdot)$, is a single-peaked function, and $r^*(\cdot)$ is a single-valued function. If $M \geq 4$, then $r^*(M) \in (0, \overline{r}_M)$ and $r^*(M)$ is characterized by,*

$$r^*(M) \ln(M) = \frac{M^{r^*(M)} + 1}{M^{r^*(M)} - 1}.$$

Further, $r^*(\cdot)$ is strictly decreasing and $\lim_{M \to \infty} r^*(M) = 0$. Finally, for any $M \geq 2$ and r such that $r \in (0, \overline{r}_{M+1})$, it holds that $X_{M+1}^*(r) > X_M^*(r)$.

We first convey the intuition behind the relationship between equilibrium total effort and the parameter r. Nti (1999) shows that the fraction of each player's rent that is dissipated in equilibrium in the asymmetric two-player Tullock contest is non-monotonic in r. Drawing on the equivalence between individual equilibrium behavior in Nti setting and our setting, consider the setting studied by Nti. For fixed prize values, when r is close to zero, the outcome of the contest is largely determined by luck. As a result, players do not have strong incentives to exert effort. As r increases, the outcome of the contest depends less on chance, and more on players' efforts. This provides stronger incentives for both players to exert effort. However, if the contest is asymmetric enough, as r increases further, the stronger player – who always exerts more effort than the weaker player – becomes ever more likely to win the contest. This discourages the weaker player from exerting effort, which may also mean that the strong player need not exert much effort in equilibrium. Combining these two effects gives rise to a non-monotonic relationship between r and total effort.

In our setting, a higher value of M corresponds to a greater degree of asymmetry in the contest; for M large enough, the value of r that maximizes total effort will be in the interior of the feasible set; i.e., $r^*(M) \in (0, \overline{r}_M)$. As M gets larger, the degree of asymmetry grows, and the discouragement effect described above is exacerbated. Lower values of r introduce a greater amount of chance into each contest, which dampens this effect by reducing the advantage of the stronger player. As a result, $r^*(\cdot)$ is decreasing.

Finally, as M gets large, the only way to prevent complete discouragement of the periphery players is by choosing r closer to zero, and so $r^*(M)$ approaches zero as M gets large.

It is also interesting to call attention to the dependency on M of total equilibrium effort (for fixed r). Note that an increase in M affects effort via two distinct channels. First, as already discussed, increasing M increases the degree of asymmetry in the contest, which is akin to increasing the prize value of the strong player in Nti's (1999) setting. Nti shows that, ceteris paribus, an increase in the strong player's prize value increases his effort, and reduces the effort of the weaker player. The net effect on total effort is ambiguous. But there's a second channel in our setting that is absent in Nti's setting; an increase in M also increases the number of periphery players. Although the individual effort of each periphery player decreases, it can be shown that the combined effort of these M periphery players is increasing in M. The net effect is that an increase in M unambiguously increases total equilibrium effort under the Tullock CSF.

We next compare the total equilibrium effort on the star under the Tullock and APA CSFs. Recall that on symmetric networks, equilibrium total effort is always greater under the APA CSF than under the Tullock CSF. Our next result shows that for star networks, the opposite conclusion may hold.

Proposition 2. *Suppose the network is a star with M contests. Fix $r = 1$, and suppose that $M \geq 3$. Then equilibrium total effort is higher under the Tullock CSF than under the APA CSF.*

The intuition for Proposition 2 closely relates to that of Proposition 1. As shown in Proposition 1, under the Tullock CSF total equilibrium effort may be decreasing in r when M is sufficiently large. But the APA CSF is just the limiting case of the Tullock CSF as $r \to \infty$. As a result, when M is sufficiently large (in this case, $M \geq 3$), so that there is a sufficient degree of asymmetry in the network, the less cutthroat competition induced under the Tullock generates greater total effort in equilibrium.

The Exclusion Principle

A rather surprising result from the contest literature is the so-called "Exclusion Principal", first demonstrated by Baye et al. (1993). The exclusion principal states that, in an asymmetric APA with at least 3 players, a contest designer may be able to increase expected total effort by excluding the player with the highest value. Intuitively, the presence of the high-value player discourages the other two players from bidding aggressively. Removing the high-value player "levels the playing field" and encourages fiercer compe-

tition amongst the remaining players. Fang (2002) shows that the Exclusion Principal does not carry over to the Tullock CSF. Intuitively, the high level of noise in the Tullock CSF means that the discouragement effect from having the strong player in the contest is somewhat mitigated. Leveling the playing field in this case does not stimulate enough effort from the remaining players to offset the drop in effort caused by the removal of he high-value player.

We next explore this idea in our network setting. To do so, we compare the expected total equilibrium effort on an M-hybrid network, with the total effort of the underlying network. We obtain an exclusion result akin to that of Baye et al under the APA CSF. In contrast to Fang's result, we show that a similar exclusion result also holds under the Tullock CSF in our setting. Before proceeding, let $x(\mathbf{g})$ denote individual expected equilibrium effort when the network is $\mathbf{g}$. Similarly, let $X(\mathbf{g})$ denote equilibrium expected total effort when the network is $\mathbf{g}$.

Theorem 3. *Let $\mathbf{g}'$ be an M-hybrid network with an underlying network, $\mathbf{g}$, summarized by $[N, d^p; M, d^c]$. Under the APA CSF, $X(\mathbf{g}) > X(\mathbf{g}')$ if and only if $N > d^c$. Under the Tullock CSF, $X(\mathbf{g}) > X(\mathbf{g}')$ if and only if*

$$N > \frac{(d^c)^2 + 1}{d^c - 1}. \tag{16}$$

Theorem 3 establishes that if the network structure is an M-hybrid, then a contest designer may be able to increase total equilibrium effort by excluding the hybrid player. Under the APA CSF, this exclusion result holds so long as there is *some* degree of asymmetry between the hybrid player and the underlying players in the network. This finding mirrors the results of Baye et al. (1993).

Our exclusion result under the Tullock CSF contrasts existing results from the contest literature (see, e.g., Fang, 2002; Matros, 2006). The condition given in (16) can be interpreted as requiring a sufficient degree of asymmetry in the hybrid network. To see this, note that (16) requires that N is sufficiently large, relative to d^c. By the link property of biregular networks, $N d^p = M d^c$, when N is large, relative to d^c it must mean that d^p is small, relative to M. That is, the hybrid player, who competes in M contests, competes for sufficiently more prizes than each player in the underlying network (each of these players compete for d^p prizes).

The impact on aggregate behavior of adding a or removing a player from a network, not only depends on that player's direct contribution, but also depends on indirect effects that stem from the player's influence on the structure of interactions (see, e.g. Ballester et al., 2006). Previous results on the Exclusion Principle in the contest literature mainly

focus on one particular pattern of interactions (one in which all players compete for a single prize); the addition/removal of one player does not affect this structure.[17] When indirect network effects are taken into account, our Theorem 3 suggests that the Exclusion Principle is a more robust phenomenon than previously thought. The following example illustrates our finding.

Example 3. *Suppose $\mathbf{g}'$ is an M-hybrid with an underlying network, $\mathbf{g}$, where $\mathbf{g}$ is summarized by $[N, d^p; M, d^c] = [6, 2; 4, 3]$. Consider the Tullock CSF with noise parameter $r = 1$, and assume $V = 1$. Under the APA CSF:*

$$X(\mathbf{g}) = MV = 4$$

and,

$$X(\mathbf{g}') = d^p V = 2.$$

Under the Tullock CSF:

$$X(\mathbf{g}) = \frac{(d^c - 1)}{d^c} MV = \frac{8}{3} \approx 2.667$$

and,

$$X(\mathbf{g}') = \frac{(2N + 1 - d^c)N}{(N + 1)^2} d^p V = \frac{120}{49} \approx 2.449.$$

Summarizing

In this section, we have shown that a number of results from the contest literature can be obtained by varying the structure of the network in our framework. Table 1 summarizes these linkages.

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[17]Two exceptions are Dahm and Esteve-Gonzalez (2014) and Dahm (2017). Both studies explore a particular network structure in which all players compete for a main prize, while a subset of disadvantaged players also compete for a secondary prize. Dahm (2017) shows that excluding an advantaged player altogether may increase total effort under the APA CSF, but even greater effort can be generated by only excluding the advantaged player from competing for the secondary prize.

Table 1: A Unified Framework

	Contest Results and Corresponding Network Structures	
Result	Related Literature	Network Structure
Behavior in Symmetric Contests	Tullock (1980), Hillman and Riley (1989), BKD	Biregular Networks
Behavior in Asymmetric Contests	Hillman and Riley (1989), BKD, Nti (1999, 2004), Stein (2002), Matros (2006)	Star Networks
Exclusion Principle	Baye et al. (1993), Fang (2002), Matros (2006), Menicucci (2006)	M-Hybrid Networks

4 Network Structures

In this section, we explore how the underlying network structure affects equilibrium behavior. Consider two biregular networks, $\mathbf{g}_1$ and $\mathbf{g}_2$, each with N players. Summarize network $\mathbf{g}_k$ by $[N, d_k^p, M_k, d_k^c]$. Our interest is in assessing how equilibrium behavior varies with the network parameters, N, d^p, M, or d^c. Let V_k denote the prize value associated with each contest when the network structure is $\mathbf{g}_k$. We wish to distinguish the impact of a change in the network structure from, say, an increase in the total value for which players compete; so, we assume that the total value of all prizes in each network is the same: $M_1 V_1 = M_2 V_2$. Note that since the link property, $N d^p = M d^c$, must hold for the network to be biregular, one can not assess a ceteris paribus change in one of the network parameters, and simultaneously preserve biregularity. Nevertheless, our next result captures some sense in which equilibrium behavior is affected by a change to the structure of the network.

Theorem 4. *Let $\mathbf{g}_1$ and $\mathbf{g}_2$ be two biregular networks with N players, such that the total value of all prizes in each network is the same: $M_1 V_1 = M_2 V_2$. Then under the Tullock CSF, the following four statements are equivalent:*

(i) $d_1^p V_1 > d_2^p V_2$

(ii) $d_1^c > d_2^c$

(iii) $X(\mathbf{g}_1) > X(\mathbf{g}_2)$

(iv) $x(\mathbf{g}_1) > x(\mathbf{g}_2)$

Under the APA CSF, $X(\mathbf{g}_1) = X(\mathbf{g}_2)$ and $x(\mathbf{g}_1) = x(\mathbf{g}_2)$.

Theorem 4 describes how total and individual equilibrium efforts are affected when the structure of the network changes, but where the number of players, and the total value of all prizes in each network is held constant.

To illustrate the result of Theorem 4, consider the two network structures in Figure 6. In each network, there are three players and three prizes. So, if each prize in both networks is worth V, then the total prize value in each network is $3V$. Note that in the circle network of Figure 6(a), $d^p = d^c = 2$; while in the complete network in Figure 6(b) $d^p = d^c = 3$. Theorem 4 implies that under the Tullock CSF, the complete network generates greater individual effort than the circle network. Consider changing the network structure from the circle network to the complete network. Equilibrium efforts are affected via two channels. First, in the complete network, each player competes for a greater total value ($3V$ as compared to $2V$); this effect encourages greater effort from each player. Second, in the complete network, each player faces more competition in each contest. Examining the expression for individual effort in Lemma 1, it is clear that an increase in the number competitors (d^c) decreases individual efforts. As it happens, the first effect always dominates the second effect under the Tullock; hence, the complete network generates greater total effort than the circle. Under the APA CSF, these two effects exactly offset one another, leading to the same total and individual expected efforts for either network structure.

Theorem 4 is also useful for providing insights into the optimal design of networks. Suppose there are a fixed number of players, N, and a contest designer with a prize budget of B. The designer chooses a biregular network, with the goal to maximize the sum of

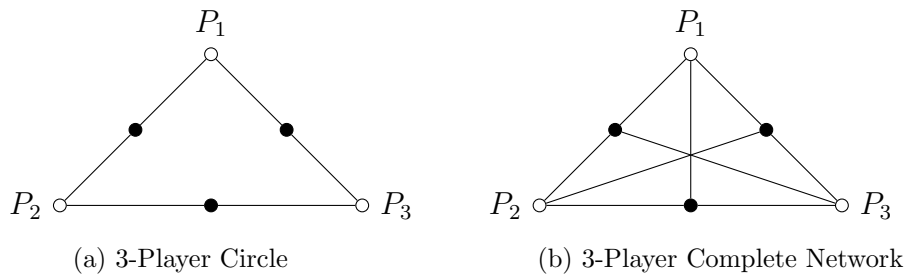(a) 3-Player Circle   (b) 3-Player Complete Network

Figure 6: Two 3-player network structures. Players are represented as hollow nodes; contests are represented as solid nodes.

equilibrium efforts. In other words, the designer takes N and B as given, and chooses d^p, M, and d^c to maximize total equilibrium effort, subject to the constraints $N d^p = M d^c$ and $MV = B$.[18] As a straightforward consequence of Theorem 4, under the Tullock CSF, the designer would be indifferent between any complete network structure. This must be true since, in general $d^c \leq N$, and for a complete network structure, $d^c = N$. Part (ii) of Theorem 4 immediately implies that this network structure would indeed maximize equilibrium efforts, subject to the aforementioned constraints. The following numerical example illustrates.

Example 4. *Let $\mathbf{g}$ be a biregular network where $N = 3$. Suppose that $B = 12$ and $r = 1$. Denote by X^* total equilibrium effort under the Tullock CSF.*

- *If $M = 1$, there is a unique biregular network, and this network is complete: $d^p = 1$ and $d^c = 3$. Total equilibrium effort is*

$$X^* = \frac{r(d^c - 1)}{d^c} MV = \frac{2}{3} B = 8.$$

- *If $M = 2$, there is a unique biregular network, and this network is complete: $d^p = 2$ and $d^c = 3$. Total equilibrium effort is*

$$X^* = \frac{r(d^c - 1)}{d^c} MV = \frac{2}{3} B = 8.$$

- *If $M = 3$, then there are two biregular networks:*

  1. *A complete network: $d^p = d^c = 3$, which yields total effort of*

$$X^* = \frac{r(d^c - 1)}{d^c} MV = \frac{2}{3} B = 8.$$

  2. *A circle network: $d^p = d^c = 2$, which yields total effort of*

$$X^* = \frac{r(d^c - 1)}{d^c} MV = \frac{1}{2} B = 6.$$

Next, suppose the number of prizes is fixed at M, and consider adding a player to a biregular network, $\mathbf{g}$. The resulting network structure, $\mathbf{g}'$ is thus a σ-hybrid. We

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[18]There is no reason to think that a contest designer should be restricted to only choosing a symmetric network structure. Yet the purpose of this exercise is to shed light on how the network structure affects equilibrium behavior.

have already shown in Lemmas 3 and 7 that, unless σ is sufficiently large, there will always exist an equilibrium in which the additional player is inactive, and there is no impact on the behavior of the players in **g**. Moreover, we showed in Theorem 3 that if this additional player competes in every contest, and the resulting network structure is asymmetric enough, then the equilibrium effort of each player in **g** will decrease, and total effort in **g'** is lower than in **g**. Finally, consider adding an additional player to a biregular network in such a way that the biregularity of the network is preserved. Since the link property, $Nd^p = Md^c$, must hold, it only make sense to answer this question for complete networks. In a complete network, $N = d^c$. By Lemma 1 it is clear that under the Tullock CSF, individual equilibrium effort decreases, while total equilibrium effort increases, following the entry of an additional player. Under the APA CSF, Lemma 5 implies that total expected equilibrium effort is unchanged, and hence individual expected efforts must fall.

5   Conclusion

In this paper we have proposed a framework for studying contests on networks. We characterized equilibrium in terms of the underlying network structure, and studied how this structure affects equilibrium behavior. Furthermore, we have shown that a number of results from the contest literature may be obtained in our framework by varying the structure of the network. In addition, we have provided a new exclusion result, akin to Baye et al.'s (1993) Exclusion Principle, but which is relevant under the Tullock CSF. This result contrasts the existing literature, and highlights the relevance of network effects in our model.

6   Appendix

Proof of Lemma 1

Let x^* be as given in the lemma. We will show that x^* is a best response to $\mathbf{x}^*_{-i} \equiv (x^*, ..., x^*)$. Let $\Gamma(x_i, \mathbf{x}_{-i}) \equiv \frac{\partial \pi_i(x_i, \mathbf{x}_{-i}, \mathbf{g})}{\partial x_i}$. First, we show that $\Gamma(x^*, \mathbf{x}^*_{-i}) = 0$ and, for $r \leq \frac{d^c}{d^c - 1}$, the second order condition is satisfied. Finally, we will show that $\pi_i(x^*, \mathbf{x}^*_{-i}) > 0$.

Since the network is biregular, if player i competes in contest m (so that $g_{im} > 0$), then $\sum_{j \neq i} g_{jm} = d^c - 1$. If player i does not compete in contest m, then $g_{im} = 0$. Hence,

for each contest, m, it holds:

$$\frac{g_{im} \sum_{j \neq i} g_{jm}}{\left(\sum_{j \in \mathcal{N}} g_{jm} \right)^2} = \frac{d^c - 1}{(d^c)^2} g_{im}.$$

Then, using (6),

$$\Gamma(x^*, \mathbf{x}^*_{-i}) = \frac{rV}{x^*} \sum_{m \in \mathcal{M}} \left\{ \frac{g_{im} \sum_{j \neq i} g_{jm}}{\left(\sum_{j \in \mathcal{N}} g_{jm} \right)^2} \right\} - 1$$

$$= \frac{rV}{x^*} \sum_{m \in \mathcal{M}} \left\{ \frac{d^c - 1}{(d^c)^2} g_{im} \right\} - 1$$

$$= \left(\frac{r(d^c - 1)}{(d^c)^2} d^p V \right) \frac{1}{x^*} - 1 = 0.$$

To show that x^* is in fact a best-reply to $\mathbf{x}^*_{-i}$ we now show that the second-order condition is satisfied, and $\pi_i(x^*, \mathbf{x}^*_{-i}, \mathbf{g}) \geq 0$. The second-order condition for player i is

$$\frac{\partial \Gamma_i(x_i, \mathbf{x}^*_{-i})}{\partial x_i} \Big|_{x_i = x^*} = \frac{(r - 1) d^c - 2r}{d^c} rV (x^*)^{2r-2} \sum_{m=1}^{M} (d^c - 1)$$

$$= \frac{(r - 1) d^c - 2r}{d^c} (d^c - 1) rMV (x^*)^{2r-2} < 0 \tag{17}$$

The SOC given in equation (17) is satisfied if and only if $(r - 1)d^c - 2r < 0$, which holds if $d^c = 2$. If $d^c > 2$ then (17) holds if and only if $r < \frac{d^c}{d^c - 2}$, which is satisfied under the condition, $r < \frac{d^c}{d^c - 1}$, given in the lemma. Next, we show that $\pi_i(x^*, \mathbf{x}^*_{-i}, \mathbf{g}) \geq 0$. Note that

$$\pi_i(x^*, \mathbf{x}^*_{-i}, \mathbf{g}) = \sum_{m=1}^{M} \left\{ \frac{g_{im} x^{*r}}{\sum_{j=1}^{N} g_{jm} x^{*r}} V \right\} - x^* = \frac{d^p V}{(d^c)^2} [d^c - r(d^c - 1)].$$

The term in square brackets on the right-hand side (RHS) of the expression above is positive if and only if $r \leq \frac{d^c}{d^c - 1}$, which establishes that x^* is a best-reply to $\mathbf{x}^*_{-i}$.

Now, we show that $(x^*, ..., x^*)$ is the unique symmetric equilibrium. Any symmetric equilibrium, $(x', ..., x')$, must satisfy $\Gamma(x', \mathbf{x}'_{-i}) = 0$. However, it is straightforward to show that $\Gamma(x, x, ..., x)$ is strictly decreasing in x. Hence, there is a unique solution to $\Gamma(x', \mathbf{x}'_{-i}) = 0$, which, as shown, is given by x^*. $\qquad \square$

Proof of Lemma 2

Suppose the network is a star with N periphery players. The first-order condition for each periphery player $i \in \{1, \ldots, M\}$ is given in equation (7), and the first-order condition for the central player, $M + 1$, is given in (8).

Equations (7) and (8) together imply:

$$x_{M+1} = \sum_{i=1}^{M} x_i. \tag{18}$$

Suppose that each periphery player i adopts a symmetric strategy: $x_i = x$. Then (18) implies that $x_{M+1} = Mx$. Using equation (7) it is easily verified that

$$x = x^* = \frac{M^r}{(1 + M^r)^2} rV.$$

Next, we confirm that the second-order conditions are satisfied. Note that for a star network, the payoff to each periphery player only depends directly on the effort of the central player. So, let $\pi_i(x, x_{M+1})$ denote the payoff to a periphery player, i, when she chooses effort, x, and the central player chooses x_{M+1}. It may be verified that the second-order sufficient condition for periphery player i, $\frac{\partial^2 \pi_i(x_i, x^*_{M+1})}{\partial x_i^2}\big|_{x_i = x^*} < 0$, is satisfied if and only if $rM^r < M^r + 1 + r$, which is implied by the condition $rM^r \leq M^r + 1$, given in the lemma. It may be verified that the second-order condition for the central player is satisfied for any $M \geq 1$.

Finally, we show that $\pi_i(x^*, Mx^*) \geq 0$ if and only if $rM^r \leq M^r + 1$. Note that:

$$\pi_i(x^*, Mx^*) = \frac{x^{*r}}{x^{*r} + (Mx^*)^r} V - x^* = \frac{(1 + (1 - r) M^r)}{(1 + M^r)^2} V.$$

Thus, $\pi_i(x^*, Mx^*) \geq 0$ if and only if $rM^r \leq M^r + 1$. Note that since each periphery player earns a non-negative payoff, the central player also earns a non-negative payoff. $\square$

Proof of Lemma 3

Let $\mathbf{g}'$ be a 1-hybrid with an underlying network summarized by $[N, d^p; M, d^c]$. Assume, without loss of generality, that the hybrid player, $N+1$ competes in contest 1. To establish the lemma, it suffices to show that when each player $i \in \{1, \ldots, N\}$ chooses effort x^*, given in the lemma, then player $N + 1$ does not have a profitable deviation.

Suppose that each player $i \in \{1, \ldots, N\}$ chooses effort x^*. Note that the total effort expended to each contest by these players is $d^c x^* = \frac{d^c - 1}{d^c} d^p V$. Then, see that for any

$d^c \geq 2$ and $d^p \geq 2$ it holds that $d^c x^* = \frac{d^c - 1}{d^c} d^p V \geq V$. Suppose player $N+1$ chooses effort $x \in \mathbb{R}_+$; the expected payoff to player $N+1$ is:

$$\pi_{N+1}(x, \mathbf{x}^*, \mathbf{g}') = \frac{x}{x + \sum_{j=1}^{N} g_{j1} x_j} V - x$$

$$= \frac{x}{x + d^c x^*} V - x$$

$$\leq x \left(\frac{V}{x + V} - 1 \right)$$

$$\leq 0$$

The above inequality is strict if $x > 0$. Hence, the only best response for player $N+1$ is to choose $x = 0$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Proof of Lemma 4

Consider some underlying player $i \in \{1, \ldots, N\}$. Suppose that the underlying players other than i choose efforts according to, $\mathbf{x}_{-i} = (x, \ldots, x)$, where $x > 0$, while the hybrid player chooses x_{N+1}. If player i chooses effort, x_i, her expected payoff is:

$$\pi_i(x_i, \mathbf{x}_{-i}, x_{N+1}) = \sum_{m=1}^{M} \frac{g_{im} x_i}{(d^c - 1)x + x_i + x_{N+1}} V - x_i.$$

The first order condition is

$$\frac{\partial \pi_i(x_i, \mathbf{x}_{-i}, x_{N+1})}{\partial x_i} = \sum_{m=1}^{M} \frac{(d^c - 1)x + x_{N+1}}{((d^c - 1)x + x_i + x_{N+1})^2} g_{im} V - 1 = 0.$$

In a symmetric equilibrium, each underlying player chooses the same effort, $x_j = x^*$ for $j \in \{1, \ldots, N\}$. The first-order condition becomes,

$$\sum_{m=1}^{M} \frac{(d^c - 1)x^* + x_{N+1}}{(d^c x^* + x_{N+1})^2} g_{im} V = \frac{(g - 1)x^* + x_{N+1}}{(d^c x^* + x_{N+1})^2} d^p V = 1. \qquad (19)$$

In a symmetric equilibrium, the payoff to the central player $N+1$ is:

$$\pi_{N+1}(x_{N+1}, \mathbf{x}^*) = \frac{x_{N+1}}{d^c x^* + x_{N+1}} M V - x_{N+1}.$$

The first order condition is

$$\frac{d^c x^*}{(d^c x^* + x_{N+1}^*)^2} MV = 1. \tag{20}$$

From (19) and (20), we get

$$x_{N+1}^* = \frac{d^c(M - d^p) + d^p}{d^p} x^* = (N + 1 - d^c)x^*. \tag{21}$$

From (4) and (21),

$$x_{N+1}^* = (N + 1 - g)x^*. \tag{22}$$

Finally, (19) and (22) give

$$x^* = \frac{N}{(N+1)^2} d^p V.$$

It is easy to verify that the second-order conditions are satisfied for all players. More-over, given efforts, $x_i = x^*$ for all $i \in \{1, \ldots, N\}$ and $x_{N+1}^* = (N + 1 - d^c)x^*$, it is easy to check that the expected payoffs to all players are positive. $\qquad\square$

Proof of Lemma 5

First, we show that there is a symmetric equilibrium in which each player chooses their effort according to the distribution function F^* (where F^* is given in (10)), with support $[0, d^p V]$. Let $\pi_i(x, F_{-i}^*)$ denote the payoff to player i from choosing effort $x \in [0, d^p V]$ when all other players choose their efforts according to F^*. See that

$$\pi_i(x, F_{-i}^*) = \sum_{m \in \mathcal{M}} \left(g_{im} V F^*(x)^{\sum_{j \neq i} g_{jm}} \right) - x.$$

Since the network is biregular, if $g_{im} = 1$ for some m then $\sum_{j \neq i} g_{jm} = d^c - 1$. Hence,

$$g_{im} F^*(x)^{\sum_{j \neq i} g_{jm}} = g_{im} F^*(x)^{d^c - 1}$$

So,

$$\pi_i(x, F_{-i}^*) = F^*(x)^{d^c - 1} \sum_{m \in \mathcal{M}} g_{im} V - x$$

$$= F^*(x)^{d^c - 1} (d^p V) - x = 0,$$

Thus, any $x \in [0, d^p V]$ is a best response of player i to F_{-i}^*. Therefore, there exists

a symmetric equilibrium in which each player randomizes continuously on the interval $[0, d^p V]$ according to the distribution function given by (10).

The arguments that establish that this is the *unique* symmetric equilibrium follow similar arguments to those made by BKD in the proof of their Theorem 1. In particular, their arguments may be used to establish the following facts about any symmetric equilibrium:

(i) All players randomize continuously on $[0, d^p V]$, with strictly increasing CDFs over this interval.

(ii) Each player earns an expected payoff of 0.

The facts above may be used to establish that there is a unique symmetric equilibrium. Next, see that the expected effort of each player i in the symmetric equilibrium is

$$E[x^*] = \frac{1}{d^c - 1} \left(\frac{1}{d^p V} \right)^{\frac{1}{d^c - 1}} \int_0^{d^p V} x^{\frac{1}{d^c - 1}} dx = \frac{d^p V}{g}.$$

The expected total equilibrium effort is

$$E[X^*] = N \times E[x^*] = \frac{N d^p}{d^c} V = MV.$$

$\square$

Proof of Lemma 6

First we show that the strategies described in the lemma constitute an equilibrium. So, suppose the periphery players, $\{1, \ldots, M\}$, bid according to $\mathbf{F}^* = (F^*, \ldots, F^*)$, where F^* is given in the lemma. If the central player, $M+1$, chooses $x \in [0, V]$, his expected payoff is:

$$\pi_{M+1}(x, \mathbf{F}^*) = V \sum_{m=1}^M F^*(x) - x = V \left(M - 1 + \frac{x}{V} \right) - x = (M-1)V.$$

Hence, the central player is indifferent among all $x \in [0, V]$ and uniform randomization on $[0, V]$ is a best-response to $\mathbf{F}^*$. Now suppose the central player bids uniformly on $[0, V]$; let G^* denote this CDF: $G^*(x) = \frac{x}{V}, x \in [0, V]$. If periphery player i, chooses effort, $x \in [0, V]$, then

$$\pi_i(x, G^*) = \left(\frac{x}{V} \right) V - x = 0$$

Hence, F^* is a best response for player i.

The arguments that establish that this is the *unique* symmetric equilibrium follow similar arguments to those made by BKD in the proof of their Theorem 2. In particular, their arguments may be used to establish the following facts about any symmetric equilibrium:

(i) The central player randomizes continuously on $[0, V]$, with a strictly increasing CDF over this interval.

(ii) The periphery players follow mixed strategies with support in $[0, V]$ and place no mass at any $x \in (0, V]$.

(iii) The periphery players randomize continuously on $(0, V]$, with strictly increasing CDFs over this interval.

(iv) The expected payoff of the central player is $(M - 1)V$, while the expected payoff of each periphery player is 0.

The facts above may then be applied to establish that there is a unique symmetric equilibrium.

Now, in the symmetric equilibrium, the central player chooses effort uniformly on $[0, V]$; so his expected effort is $E[x_{M+1}] = \frac{V}{2}$. The sum of the expected efforts exerted by the periphery players is

$$\sum_{i=1}^{M} E[x_i] = ME[x_i] = M \int_0^V x \, dG^*(x) = M \int_0^V x \left(\frac{V}{M}\right) dx = \frac{V}{2}.$$

Hence, expected equilibrium total effort is $X^* = \frac{V}{2} + \frac{V}{2} = V$. $\qquad\qquad\square$

Proof of Lemma 7

Let $\mathbf{g}'$ be a 1-hybrid with underlying network $\mathbf{g}$ summarized by $[N, d^p; M, d^c]$ with $d^p \geq 2$. Let F^* be as given in the lemma. It suffices to show that when the underlying players bid according to $\mathbf{F}^* = (F^*, \ldots, F^*)$, then the hybrid player's expected payoff is no greater than zero: That is, for any $x \in \mathbb{R}_+$, we will show $\pi_{N+1}(x, \mathbf{F}^*, \mathbf{g}') \leq 0$. First note that we may restrict attention to $x \leq V$, as any $x > V$ guarantees the hybrid player a payoff less than zero. Also note that $0 \leq x \leq V < d^p V$ implies $F^*(x) = \left(\frac{x}{d^p V}\right)^{\frac{1}{d^c - 1}}$. Thus, we have

$$\pi_{N+1}(x, \mathbf{F}^*, \mathbf{g}') = F^*(x)^{d^c}V - x$$

$$= \left(\frac{x}{d^pV}\right)^{\frac{d^c}{d^c-1}} V - x$$

$$< \left(\frac{x}{d^pV}\right) V - x$$

$$= \frac{x}{d^p} - x$$

$$\leq 0$$

The first inequality holds since $\frac{x}{d^pV} < 1$, and $\frac{d^c}{d^c-1} > 1$. The final inequality holds since, by assumption, $d^p \geq 2$, and $x \geq 0$. □

Proof of Lemma 8

We first show that the strategies given in the lemma constitute a symmetric equilibrium. Suppose the underlying players, $\{1, \ldots, N\}$, randomize according to $\mathbf{F}^* = (F^*, \ldots, F^*)$, where F^* is as given in the lemma. If the hybrid player, $N + 1$, bids $x \in [0, d^pV]$, then his expected payoff is

$$\pi_{N+1}(x, \mathbf{F}^*) = \sum_{m=1}^{M} VF^*(x)^{d^c} - x = (M - d^p)V.$$

Therefore, the hybrid player is indifferent among all $x \in [0, d^pV]$. Hence the mixed strategy outlined in the lemma for the hybrid player is a best response to $\mathbf{F}^*$. Next, suppose that the underlying players, other than underlying player i, randomize according to $\mathbf{F}^*_{-i} = (F^*, \ldots, F^*)$ and the hybrid player randomizes according to G^*, where G^* is as given in the lemma. Then, if underlying player i chooses $x \in [0, d^pV]$, her expected payoff is:

$$\pi_i(x, \mathbf{F}^*_{-i}, G^*) = \sum_{m=1}^{M} g_{im} VF^*(x)^{d^c-1}G^*(x) - x = 0.$$

Therefore, player i is indifferent among all $x \in [0, d^pV]$. Hence, the mixed strategy outlined in the lemma for player i is a best response to $\mathbf{F}^*_{-i}$ and G^*.

The expected equilibrium effort of each underlying player is

$$E[x_i] = \int_0^{d^pV} x \, dF^*(x) = d^pV - \int_0^{d^pV} F^*(x)dx.$$

Similarly, the expected equilibrium effort of the hybrid player is

$$E[x_{N+1}] = \int_0^{d^pV} x dG^*(x) = d^pV - \int_0^{d^pV} G^*(x)dx$$

$$= d^pV - \frac{1}{d^pV} \int_0^{d^pV} x \left(F^*(x)\right)^{1-d^c} dx$$

$$= d^pV - \frac{d^cM}{d^p} \left(d^pV - \int_0^{d^pV} F^*(x)dx\right)$$

Therefore, total expected equilibrium effort is

$$X(\mathbf{g'}) = N \times E[x_i] + E[x_{M+1}]$$

$$= Nd^pV - N \int_0^{d^pV} F^*(x)dx + d^pV - d^cMV + \frac{d^cM}{d^p} \int_0^{d^pV} F^*(x)dx$$

$$= d^pV$$

Where the final equality is obtained by applying the link property (4): $Nd^p = Md^c$.

The arguments that establish that this is the *unique* symmetric equilibrium follow similar arguments to those made by BKD in the proof of their Theorem 2. In particular, their arguments may be used to establish the following facts about any symmetric equilibrium:

(i) The hybrid player randomizes continuously on $[0, d^pV]$, with a strictly increasing CDF over this interval.

(ii) The underlying players follow mixed strategies with support in $[0, d^pV]$, and place no mass at any $x \in (0, d^pV]$.

(iii) The underlying players randomize continuously on $(0, d^pV]$, with strictly increasing CDFs over this interval.

(iv) The expected payoff of the hybrid player is $(M - d^p)V$, while the expected payoff of each underlying player is 0.

These facts can be used to establish that there is a unique symmetric equilibrium. □

Proof of Theorem 1

For the Tullock CSF: Compare the equilibrium characterized in Lemma 1 with the equilibrium characterized by Tullock (1980). For the APA CSF: Compare the equilibrium characterized in Lemma 5 with the symmetric equilibrium characterized by BKD (Theorem 1). $\square$

Proof of Theorem 2

For the Tullock CSF: Compare the equilibrium characterized in Lemma 2 with the equilibrium characterized by Nti (1999). For the APA CSF: Compare the equilibrium characterized in Lemma 6 with the equilibrium characterized by Proposition 2 in Hillman and Riley (1989) or Theorem 3 in BKD. $\square$

Proof of Proposition 1

If $r \leq \bar{r}_M$, then by Lemma 2 symmetric equilibrium total effort is

$$X_M^*(r) = \frac{2rM^{r+1}V}{(M^r + 1)^2}.$$

First, we show that $X_M^*(r)$ is a single-peaked function of r. Let $M \geq 2$ and note that

$$\frac{\partial X_M^*(r)}{\partial r} = (r \ln M + M^r - M^r r \ln M + 1) \frac{2M^{r+1}V}{(M^r + 1)^3}.$$

If $\frac{\partial X_M^*(r^*)}{\partial r} = 0$ at some $r = r^*$, then

$$r^* \ln(M) = \frac{M^{r^*} + 1}{M^{r^*} - 1}.$$

Mext, note that

$$\text{sign} \left[\frac{\partial^2 X_M^*(r)}{\partial r^2} \right] = \text{sign} \left[2(1 - M^{2r}) + r \ln(M) \left(M^{2r} - 4M^r + 1 \right) \right].$$

At $r = r^*$, we get

$$\text{sign}\left[\frac{\partial^2 X_M^*(r)}{\partial r^2}|_{r=r^*}\right] = \text{sign}\left[2(1 - M^{2r^*}) + \frac{M^{r^*} + 1}{M^{r^*} - 1}\left(M^{2r^*} - 4M^{r^*} + 1\right)\right]$$

$$= \text{sign}\left[-\frac{1}{M^{r^*} - 1}\left(M^{r^*} + M^{2r^*} + M^{3r^*} + 1\right)\right] < 0, \text{ for } M \geq 2.$$

Thus, if $\frac{\partial X_M^*(r^*)}{\partial r} = 0$, then $\frac{\partial^2 X_M^*(r)}{\partial r^2}|_{r=r^*} < 0$, which means that $X_M^*(r)$ is a single-peaked function of r.

We now show that $r^*(M) < \bar{r}_M$ for all $M \geq 4$. Since $X_M^*(r)$ is a single peaked function of r, it suffices to show that $\frac{\partial X_M^*(r)}{\partial r}|_{r=\bar{r}_M} < 0$. Mote that

$$\text{sign}\left[\frac{\partial X_M^*(r)}{\partial r}|_{r=\bar{r}_M}\right] = \text{sign}\left[M^{\bar{r}_M} + 1 - \bar{r}_M(M^{\bar{r}_M} - 1)\ln(M)\right]. \tag{23}$$

If $M = 4$, then $\bar{r}_4 \approx 1.192$; plugging this into (23) for $M = 4$, one may verify that $\text{sign}\left[\frac{\partial X_M^*(r)}{\partial r}|_{r=\bar{r}_M}\right] < 0$. For $M \geq 5$ we will show that $\frac{\partial X_M^*(r)}{\partial r}|_{r=1} < 0$. Since $\bar{r}_M > 1$ for all M, and $X_M^*(\bar{r})$ is a single-peaked function of r, if $\frac{\partial X_M^*(r)}{\partial r}|_{r=1} < 0$, then $\frac{\partial X_M^*(r)}{\partial r}|_{r=\bar{r}_M} < 0$. Mote that

$$\text{sign}\left[\frac{\partial X_M^*(r)}{\partial r}|_{r=1}\right] = \text{sign}\left[M + 1 - (M - 1)\ln(M)\right]$$

and

$$[M + 1 - (M - 1)\ln(M)] < 0 \text{ for } M \geq 5.$$

Hence, for $M \geq 4$ we have $r^*(M) < \bar{r}_M$ where $r^*(M)$ is characterized by:

$$r^*(M)\ln M = \frac{M^{r^*(M)} + 1}{M^{r^*(M)} - 1}. \tag{24}$$

Although M takes on only discrete integer values, for the moment, suppose that M is a continuous variable. If M is a continuous variable then (24) defines a continuous function $\tilde{r}^*(\cdot)$; we show in this case that $\frac{d\tilde{r}^*(M)}{dM} < 0$. This finding immediately implies that $r^*(\cdot)$ is strictly decreasing, as $r^*(M)$ corresponds to the particular points on $\tilde{r}(M)$ when M is an integer. The Implicit Function Theorem implies

$$\frac{d\tilde{r}^*(M)}{dM} = -\frac{\tilde{r}^*(M)}{M\ln M} < 0.$$

Thus $r^*(\cdot)$ is strictly decreasing.

We now show that $\lim_{M\to\infty} r^*(M) = 0$. Fix $\epsilon > 0$. We must show $r^*(M) < \epsilon$ for all

M sufficiently large. Since the expression, $\frac{M^a+1}{M^a-1}$ is strictly decreasing in a, (24) implies that $r^*(M) < \epsilon$ if and only if

$$\frac{M^\epsilon + 1}{(M^\epsilon - 1) \ln M} < \epsilon. \tag{25}$$

Mote that, for any $\epsilon > 0$,

$$\lim_{M \to \infty} \frac{M^\epsilon + 1}{(M^\epsilon - 1) \ln M} = \left(\lim_{M \to \infty} \frac{M^\epsilon + 1}{M^\epsilon - 1} \right) \left(\lim_{M \to \infty} \frac{1}{\ln M} \right) = 1 \times 0 = 0.$$

Thus, for M sufficiently large inequality (25) holds, which implies that $r^*(M) < \epsilon$ for M sufficiently large.

Mext we show that $X_M^*(r)$ is increasing in M, for fixed r. Let $M \geq 2$ be given. Fix $0 < r < \bar{r}_{M+1} < \bar{r}_M$; we show that $X_{M+1}^*(r) > X_M^*(r)$. To establish the result, we will show that for all $m \in [M, M+1]$ it holds that $\frac{\partial X_m^*(r)}{\partial m} > 0$. Mote that

$$\frac{\partial X_m^*(r)}{\partial m} = \frac{2rVn^r}{(m^r + 1)^3} \left[r + 1 + m^r - rm^r \right].$$

Let $\Gamma(m) = r + 1 + m^r - rn^r$. To establish the result, we must show $\Gamma(m) > 0$ for all $m \in [M, M+1]$. Mote, however, that $\Gamma(\cdot)$ is strictly decreasing, so the result follows if $\Gamma(M+1) > 0$. Recall the definition of $\bar{r}_{M+1}$:

$$(\bar{r}_{M+1} - 1)(M+1)^{\bar{r}_{M+1}} = 1.$$

Mow recall that $\bar{r}_{M+1} > 1$ for any M; so $r \leq 1$ implies $(M+1)^r(r-1) \leq 0 < (M+1)^{\bar{r}_{M+1}}(\bar{r}_{M+1} - 1)$. For all $r > 1$, the term, $(M+1)^r(r-1)$, is strictly increasing in r, so $1 < r < \bar{r}_{M+1}$ implies $(M+1)^r(r-1) < (M+1)^{\bar{r}_{M+1}}(\bar{r}_{M+1} - 1)$. Thus, we have the following:

$$\begin{aligned}
\Gamma(M+1) &= r + 1 - (M+1)^r(r-1) \\
&> r + 1 - (M+1)^{\bar{r}_{M+1}}(\bar{r}_{M+1} - 1) \\
&= r + 1 - 1 \\
&= r > 0
\end{aligned}$$

The first strict inequality follows since, as shown above, $(M+1)^r(r-1) < (M+1)^{\bar{r}_{M+1}}(\bar{r}_{M+1} - 1)$. The second equality follows by the definition of $\bar{r}_{M+1}$. Thus, $\Gamma(M+1) > 0$. $\square$

Proof of Proposition 2

By Lemma 6, it suffices to show that when $r = 1$, and $M \geq 3$ equilibrium total effort under the Tullock CSF is greater than V. Let X_M^T denote equilibrium total effort under the Tullock when $r = 1$ and there are M periphery players. Since X_M^T is strictly increasing in N (by proposition 1), we need only show $X_3^T > V$. By Lemma 2:

$$X_3^T = \frac{2M^2}{(M+1)^2} V = \frac{9}{8} V > V.$$

□

Proof of Theorem 3

Under the APA CSF, it follows immediately from Lemmas 5 and 8 that $X(\mathbf{g}) > X(\mathbf{g}')$ if and only if $M > d^p$. But the link property, (4), implies that $M > d^p$ if and only if $N > d^c$. Under the Tullock CSF, from Lemma 4, total equilibrium effort on the hybrid network, $\mathbf{g}'$, is

$$X(\mathbf{g}') = Nx^* + x_{N+1}^* = \frac{(2N+1-d^c)N}{(N+1)^2} (d^p V) = \frac{(2N+1-d^c)d^c}{(N+1)^2} MV.$$

Using Lemma 1, total equilibrium effort on the underlying network, $\mathbf{g}$, is:

$$X(\mathbf{g}) = \frac{d^c - 1}{d^c} MV.$$

So, $X(\mathbf{g}) > X(\mathbf{g}')$ if and only if:

$$\frac{d^c - 1}{d^c} > \frac{(2N+1-d^c)d^c}{(N+1)^2},$$

re-arranging,

$$\left(-N + Nd^c - (d^c)^2 - 1 \right) (N - d^c + 1) > 0.$$

Since $N \geq d^c$, the inequality above is equivalent to

$$\left(-N + Nd^c - (d^c)^2 - 1 \right) > 0,$$

or

$$N > \frac{(d^c)^2 + 1}{d^c - 1}.$$

$\square$

Proof of Theorem 4

For $k = 1, 2$ let $X(\mathbf{g}_k) = X_k$ and $x(\mathbf{g}_k) = x_k$. Since both networks have N players, clearly $X_1 > X_2$ if and only if $x_1 > x_2$. First suppose the CSF is the Tullock; using the characterization of equilibrium in Lemma 1, $X_1 > X_2$ if and only if $\frac{rd_1^c}{d_1^c-1}M_1V_1 > \frac{rd_2^c}{d_2^c-1}M_2V_2$. But, by assumption, $M_1V_1 = M_2V_2$. So, $X_1 > X_2$ if and only if $\frac{d_1^c}{d_1^c-1} > \frac{d_2^c}{d_2^c-1}$, which holds if and only if $d_1^c > d_2^c$. Thus, (iv) is equivalent to (iii), and (iii) is equivalent to (ii).

We now show (i) is equivalent to (ii). The link property for biregular networks gives, for each $k = 1, 2$: $Nd_k^p = M_kd_k^c$. Combining the link property with the assumption, $M_1V_1 = M_2V_2$, yields $\frac{Nd_1^pV_1}{d_1^c} = \frac{Nd_2^pV_2}{d_2^c}$; and hence $\frac{d_1^pV_1}{d_2^pV_2} = \frac{d_1^c}{d_2^c}$, which means $d_1^c > d_2^c$ if and only if $d_1^pV_1 > d_2^pV_2$.

Finally, by Lemma 5, under the APA CSF expected equilibrium total effort in contest network $k = 1, 2$ is M_kV_k. Clearly, each network generates the same total effort since, by assumption, $M_1V_1 = M_2V_2$. Since $X_1 = X_2$, and by assumption each network has N players, we have $x_1 = \frac{X_1}{N} = \frac{X_2}{N} = x_2$. $\square$

References

Alcalde, J. and Dahm, M. (2010). Rent seeking and rent dissipation: A neutrality result. *Journal of Public Economics*, 94(1-2):1–7.

Ballester, C., Armengol, A. C., and Zenou, Y. (2006). Who's who in networks. Wanted: The key player. *Econometrica*, 74(5):1403–1417.

Baye, M. R. and Hoppe, H. C. (2003). The strategic equivalence of rent-seeking, innovation, and patent-race games. *Games and Economic Behavior*, 44(2):217–226.

Baye, M. R., Kovenock, D., and De Vries, C. G. (1993). Rigging the lobbying process: An application of the all-pay auction. *The American Economic Review*, 83(1):289–294.

Baye, M. R., Kovenock, D., and De Vries, C. G. (1994). The solution to the tullock rent-seeking game when $R > 2$: Mixed-strategy equilibria and mean dissipation rates. *Public Choice*.

Baye, M. R., Kovenock, D., and de Vries, C. G. (1996). The all-pay auction with complete information. *Economic Theory*, 8(2):291–305.

Bramoullé, Y. and Kranton, R. (2007). Public goods in networks. *Journal of Economic Theory*, 135(1):478–494.

Bramoullé, Y. and Kranton, R. (2016). Games played on networks. In Bramoullé, Yann, A. G. and Rogers, B., editors, *The Oxford Handbook on the Economics of Networks*. Oxford University Press, Oxford.

Bramoullé, Y., Kranton, R., and D'Amours, M. (2014). Strategic interaction and networks. *The American Economic Review*, 104(3):898–930.

Calvó-Armengol, A. (2004). Job contact networks. *Journal of Economic Theory*, 115(1):191–206.

Calvó-Armengol, A. and Jackson, M. O. (2004). The effects of social networks on employment and inequality. *The American Economic Review*, 94(3):426–454.

Calvó-Armengol, A. and Zenou, Y. (2004). Social networks and crime decisions: The role of social structure in facilitating delinquent behavior. *International Economic Review*, 45(3):939–958.

Che, Y.-K. and Gale, I. (2003). Optimal design of research contests. *The American Economic Review*, 93(3):646–671.

Congleton, R., Hillman, A., and Konrad, K. (2008). *Forty years of rent-seeking research*, volume 1-2. Springer & Verlag, Heidelberg.

Dahm, M. (2017). All-pay auctions with extra prize: A partial exclusion principle. CeDEx Discussion Paper No. 2017-01.

Dahm, M. and Esteve-Gonzalez, P. (2014). Affirmative action through extra prizes. CeDEx Discussion Paper No. 2014-08.

Ewerhart, C. (2015). Mixed equilibria in Tullock contests. *Economic Theory*, 60(1):59–71.

Fang, H. (2002). Lottery versus all-pay auction models of lobbying. *Public Choice*, 112(3-4):351–371.

Franke, J. and Öztürk, T. (2015). Conflict networks. *Journal of Public Economics*, 126:104–113.

Galeotti, A., Goyal, S., Jackson, M. O., Vega-Redondo, F., and Yariv, L. (2010). Network games. *The Review of Economic Studies*, 77(1):218–244.

Gassmann, O. and Von Zedtwitz, M. (1999). New concepts and trends in international R&D organization. *Research Policy*, 28(2-3):231–250.

Goyal, S. and Joshi, S. (2003). Networks of collaboration in oligopoly. *Games and Economic Behavior*, 43(1):57–85.

Goyal, S. and Moraga-Gonzalez, J. L. (2001). R&D networks. *RAND Journal of Economics*, 32(4):686–707.

Grandjean, G., Tellone, D., and Vergrote, W. (2016). Cooperation, competition and entry in a Tullock contest. CORE Discussion Paper 2016/32.

Hillman, A. L. and Riley, J. G. (1989). Politically contestable rents and transfers. *Economics & Politics*, 1(1):17–39.

Jackson, M. O. (2008). *Social and Economic Networks*. Princeton University Press, Princeton.

Jackson, M. O. and Yariv, L. (2007). Diffusion of behavior and equilibrium properties in network games. *The American Economic Review*, 97(2):92–98.

Jackson, M. O. and Zenou, Y. (2014). Games on networks. In Young, P. and Zamir, S., editors, *Handbook of Game Theory*, volume 4. Elsevier.

König, M. D., Rohner, D., Thoenig, M., and Zilibotti, F. (2015). Networks in conflict: Theory and evidence from the great war of Africa. HICN Working Paper 195.

Konrad, K. (2009). *Strategy and Dynamics in Contests*. Oxford University Press, New York, NY.

Kovenock, D. and Roberson, B. (2015). The optimal defense of network connectivity. CESifo Working Paper No. 5653.

Marinucci, M. and Vergote, W. (2011). Endogenous network formation in patent contests and its role as a barrier to entry. *The Journal of Industrial Economics*, 59(4):529–551.

Matros, A. (2006). Rent-seeking with asymmetric valuations: Addition or deletion of a player. *Public Choice*, 129(3-4):369–380.

Menicucci, D. (2006). Banning bidders from all-pay auctions. *Economic Theory*, 29(1):89–94.

Nitzan, S. (1994). Modelling rent-seeking contests. *European Journal of Political Economy*, 10(1):41–60.

Nti, K. O. (1999). Rent-seeking with asymmetric valuations. *Public Choice*, 98(3-4):415–430.

Nti, K. O. (2004). Maximum efforts in contests with asymmetric valuations. *European Journal of Political Economy*, 20(4):1059–1066.

Stein, W. E. (2002). Asymmetric Rent-Seeking with More than Two Contestants. *Public Choice*, 113(3-4):325–336.

Tullock, G. (1980). Efficient rent-seeking. In Buchanan, J. M., Tollison, R. D., and Tullock, G., editors, *Toward a theory of the rent-seeking society*, number 4, pages 97–112. Texas A&M University Press, College Station, Texas.